Digital Twin for the human cornea: curvature estimation

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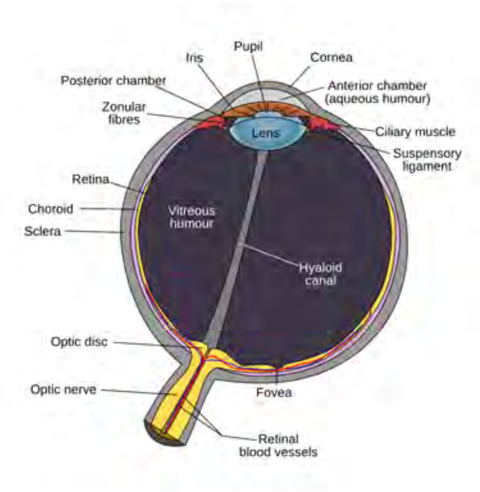
Eye's anatomy



Typical corneal dimensions:

- size 24mm,
- diameter 11.5mm,
- thickness 0.5 – 0.7mm,
- height about 2mm.

Five layers of the cornea:

- epithelium,
- Bowman's layer,
- stroma
  (90% corneal thickness),
- Descement's membrane,
- endothelium.

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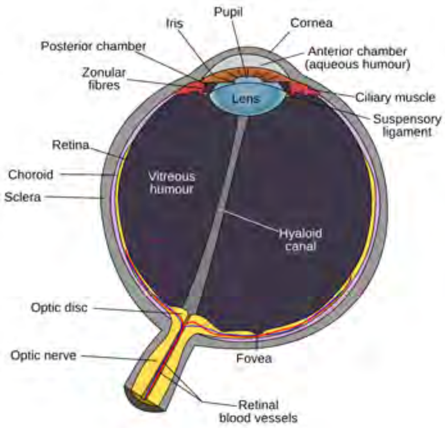

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Cornea is responsible for about $\frac{2}{3}$ of refractive power!

Importance of corneal measurements

- Ophthalmologists measure various kinds of parameters
  - Topography (tomography) (shape) of the anterior and posterior surface
  - Corneal thinkness
  - Stress and strain
  - Corneal curvature (closely related with corneal power)
- Measurements have many purposes
  - Diagnosing diseases (for ex. keratokonus, myopia, astigmatism)
  - Preparation for refractive surgery
  - Contact lens fitting
  - Scanning and monitoring corneal health
- Techniques of measurement topography[1]
  - Keratometry (image reflected)
  - Keratoscopy (projecting a concentric array of (Placido) discs)
  - Scanning slits
  - Scheimpflug correction

[1] Fowler, C. W., and T. N. Dave. "Review of past and present techniques of measuring corneal topography." Ophthalmic and Physiological Optics 14.1 (1994): 49-58.

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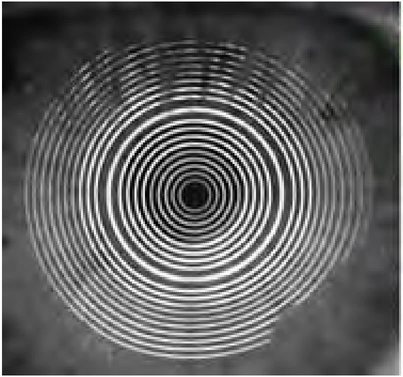

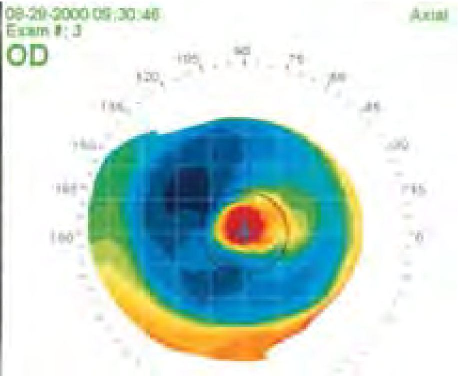

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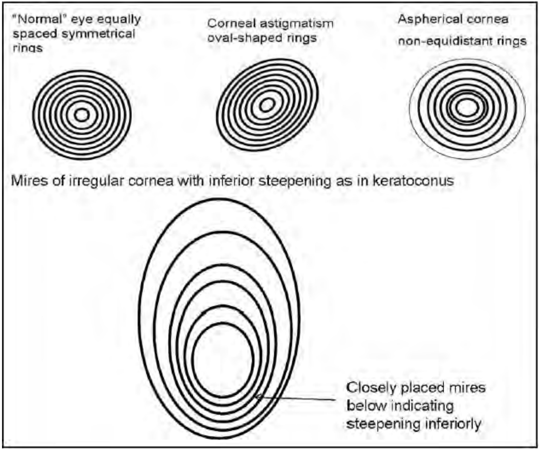

Figure: A schematic of the Placido disc keratoscopy. Picture from[2].

[2] Dharwadkar, Sachin, and B. K. Nayak. "Corneal topography and tomography." Journal of clinical ophthalmology and research 3.1 (2015): 45

Different kinds of curvature

- There are several notions of the curvature of a surface.
  - Principal curvatures.
  - Mean curvature.
  - Gaussian curvature.
- For a 2D cross section the curvature is defined as an inverse of the radius of curvature.
- Ophthalmologists use something a little bit different[2].
  - **Axial** (sagittal) radius of curvature: distance along a normal from the corneal surface to the optical axis (dependent on the reference, smooth)
  - **Tangential** (instantaneous) radius of curvature: radius of the osculating circle (the "true" radius, sensitive to the noise).
- Everything is expressed in terms of corneal power measured in **dioptres**

$$D = \frac{\text{refractive index of the cornea } - 1}{\text{radius of curvature in meters}} = \frac{1.3375 - 1}{R}.$$

Different kinds of curvature

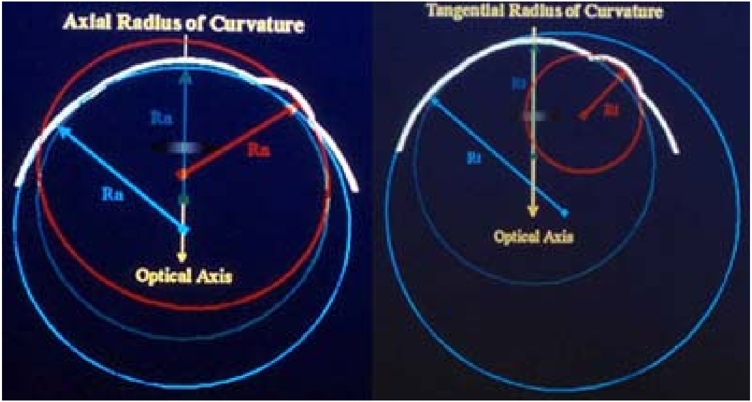

Figure: Definitions of axial and tangential radii of curvature.

Different kinds of curvature

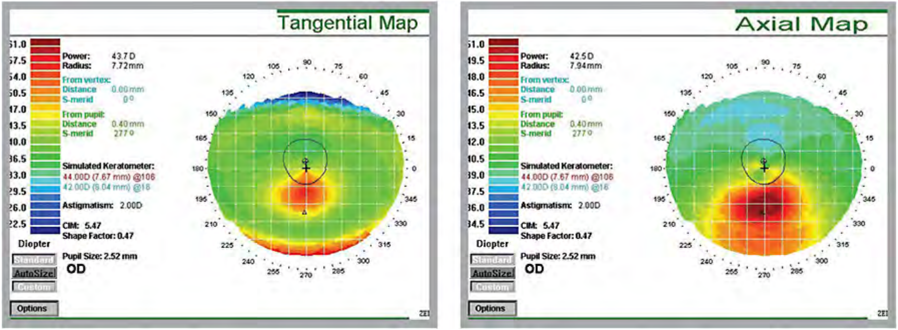

Figure: Exemplary views of radii of curvature.

The idea of a Digital Twin of the cornea

Digital Twin

A digital counterpart (copy) of a real-world object.

- A patient can have theirs cornea scanned and measured.
- The **digital twin** can be constructed based on these measurements.
- The twin can be updated by subsequent measurements.
- The twin can be used in computer simulations
  - Optical: vision, ray tracing
  - Mechanical: surgeries, deformations
  - And a **mix** of these two...
- In this way many medical procedures can be tested **noninvasively**[3,4]

[3] Cano, Daniel, Sergio Barbero, and Susana Marcos. "Comparison of real and computer-simulated outcomes of LASIK refractive surgery." JOSA A 21.6 (2004): 926-936.

[4] Kim, Youngjun, Hannah Kim, and Yong Oock Kim. "Virtual reality and augmented reality in plastic surgery: a review." Archives of plastic surgery 44.3 (2017): 179.

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Of course, this idea may be idyllic but let us start from the beginning.

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Corneal topography models

- The simplest: based on conical curves (mostly parabolas and ellipses) (Helmholtz, 1924)
- Complex: based on shell theory or FEM.
- Models based on Zernike orthogonal polynomials (1934) - describe abberation.
- Real, physical eye models.
- In a series of papers we have proposed a physical model of **intermediate complexity**[5,6]
  - Cornea is a thin membrane (constant tension and no bending moments).
  - Balance of forces: surface tension, elasticity and originating from **intraocular pressure**.
  - Let $h = h(\mathbf{x})$ be the elevation of the cornea over some reference plane.

$$-T\nabla\cdot\left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right)+kh=\frac{P}{\sqrt{1+|\nabla h|^2}},\quad h|_{\partial\Omega}=0.$$

[5] Płociniczak, Ł., Okrasiński, W., Nieto, J. J., Dominguez, O. (2014). On a nonlinear boundary value problem modeling corneal shape. Journal of Mathematical Analysis and Applications, 414(1), 461-471.

[6] Płociniczak, Ł., Griffiths, G. W., Schiesser, W. E. (2014). ODE/PDE analysis of corn

Our model

In nondimensional terms we have a prescribed mean-curvature equation with $a := \frac{kR^2}{T}$ i $b := \frac{PR}{T}$

$$-\nabla \cdot \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) + a(\mathbf{x})h = \frac{b(\mathbf{x})}{\sqrt{1 + |\nabla h|^2}}, \quad h|_{\partial\Omega} = 0.$$

- We have proved existence and uniqueness along with various estimates of a simplified version of this equation (axisymmetric with linear first term).
- The full version was tackled by Chiarra Corsato, Colette De Coster, and Pierpaolo Omari[7].

[7] Corsato, C., De Coster, C., Omari, P. (2016). The Dirichlet problem for a prescribed anisotropic mean curvature equation: existence, uniqueness and regularity of solutions. Journal of Differential Equations, 260(5), 4572-4618.

Simple approximation

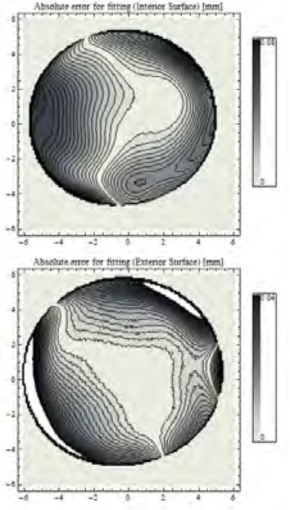

Figure: Fitting errors.

- Zeroth order approximation: small deflections $|\nabla u| \ll 1$ and axial symmetry $h = h(r)$ with $a = a_0$, $b = b_0$ constant

$$-h'' + a_0 h = b_0, \quad h'(0) = 0, \quad h(1) = 0,$$

which has a simple solution

$$h_0(r) := \frac{b_0}{a_0}\left(1 - \frac{\cosh \sqrt{a_0}\, r}{\cosh \sqrt{a_0}}\right)$$

- The mean **fitting error** is 1.4% and 3.6% for, respectively, anterior and posterior surface. Here, a_0, $b_0 \approx 2.0 - 2.5$.
- Crude estimate on the corneal deflection

$$|\nabla h| \approx \frac{\text{2 mm in height}}{\text{11 mm in width}} = 0.18 \ll 1.$$

Determination of coefficients

- Since the zeroth order approximation gives good results we can put

$$a(\mathbf{x}) = a_0 + \alpha(r,\varphi), \quad b(\mathbf{x}) = b_0 + \beta(r,\varphi).$$

- We then have $a_0 h_0 = b_0 + h_0''$ and

$$a_0 h = a_0 h_0 + a_0(h - h_0) = b_0 + h_0'' + a_0(h - h_0).$$

- Plugging the above into the main equation lets us to write

$$\boxed{\begin{aligned}\alpha(\mathbf{x})h - \frac{\beta(\mathbf{x})}{\sqrt{1+|\nabla h|^2}} = \\ - a_0 \underbrace{(h - h_0)}_{\text{shape}} + b_0 \underbrace{\left(\frac{1}{\sqrt{1+|\nabla h|^2}} - 1\right)}_{\text{deflection}} + \underbrace{\nabla \cdot \left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right)}_{\text{mean curvature}} - h_0''.\end{aligned}}$$

- **Direct problem**: provided α and β find h (not the thing that we want).

Determination of coefficients

Well- and ill-posed problems (Hadamard)

1. Have a solution (**existence**).
2. The solution is unique (**uniqueness**).
3. Small perturbations in data produce small changes in the solution (**stability**).

If a problem is not well-posed it is **ill-posed**.

- **Inverse problem**: provided h find α and β.
- Coefficients $a(\mathbf{x})$ and $b(\mathbf{x})$ encode fine structure **information** about the cornea (material properties, intra-ocular pressure).
- It is the topography $h(\mathbf{x})$ that we **can measure**.

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But we always measure with an error.

Classical example of ill-posedness

- We measure the topography with an error δ

$$\text{Typically } \delta \approx 10^{-3} \text{ mm.}$$

- Therefore, we always measure h^δ rather than h.
- As an illustrative example consider determination of the cosine of the angle of normal to the cornea ψ (one of the constituents of the equation for α and β)

$$\cos\psi = \frac{1}{\sqrt{1+|\nabla h|^2}}, \quad |\nabla h|^2 = h_r^2 + \frac{1}{r^2} h_\varphi^2.$$

- Suppose that the real cornea is almost axisymmetric

$$h^\delta(r,\varphi) = h_0(r) + \delta(\varphi), \quad \delta(\varphi) = \epsilon \sin\frac{\varphi}{\epsilon^2}, \quad \epsilon \ll 1,$$

where for simplicity we assumed that the noise depends only on the angle.

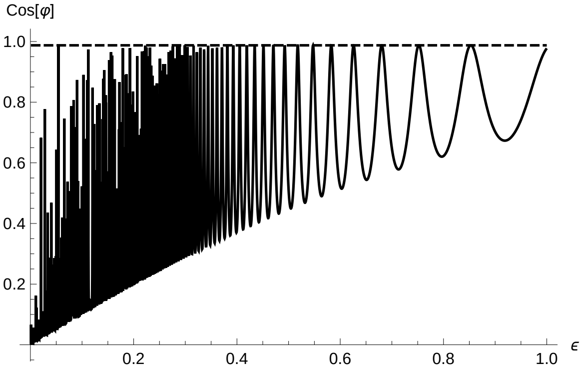
- Of course, $\delta \to 0$ for $\epsilon \to 0$.

Classical example of ill-posedness

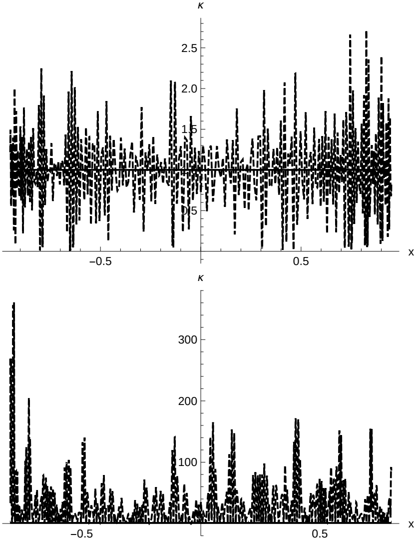
- Then,

$$\cos\psi^\delta = \frac{1}{\sqrt{1 + h_0'(r)^2 + r^{-2}\epsilon^{-2}\cos^2\frac{\varphi}{\epsilon^2}}} \le \cos\psi.$$

- Notice that $\cos\psi^\delta$ for different ϵ can take **any value** between 0 and $\cos\psi$! A **disaster**!



Curvature estimation



- Finding curvature from a noisy measurement is even more unstable!
- Suppose we would like to find the curvature of a portion of the circle (a constant).
- We can calculate this naïvely with

$$\kappa = \frac{|y''(x)|}{(1 + y'(x)^2)^{\frac{3}{2}}}.$$

- Example: circle with radius 1 and centre at $(0, 0)$. Normally distributed noise with $\delta = 10^{-5}$ (top) and $\delta = 10^{-3}$ (bottom) (visually **indistinguishable**).

- There are various methods of stable curvature estimation (essential for computer vision, in particular in medicine).
- First, **smooth** the data (but not too much) with a filter (e.x. Gaussian, moving average).
- **Least-squares** fitting of some surface (or a curve in 2D) and calculating its curvature analytically.
  - Orthogonal polynomials.
  - Second order surfaces (spheres, paraboloids etc.).
  - Splines.
  - Jets (truncated Taylor series).
- **Triangulations**.
- **M-estimation**.
- References

1. Flynn, P. J., Jain, A. K. (1989, June). On reliable curvature estimation. In CVPR (Vol. 88, pp. 5-9).
2. Worring, M., Smeulders, A. W. (1993). Digital curvature estimation. CVGIP: Image understanding, 58(3), 366-382.
3. Kalogerakis, E., Simari, P., Nowrouzezahrai, D., Singh, K. (2007). Robust statistical estimation of curvature on discretized surfaces. In Symposium on Geometry Processing (Vol. 13, pp. 110-114).

Initial results for cornea

- We have applied the method of fitting a circle to the axisymmetric cornea that was fitted to the real data with $a_0 = 1.97$ and $b = 2.27$.
- The noise level of the equipment is $\delta = 10^{-3}$. The data is smoothed with moving average.
- The curvature at r is estimated by fitting a circle to the points in its neighbourhood of width w.
- The outcome is smoothed once again.
- We conduct similar computations for $\cos\psi$.

Initial results for cornea

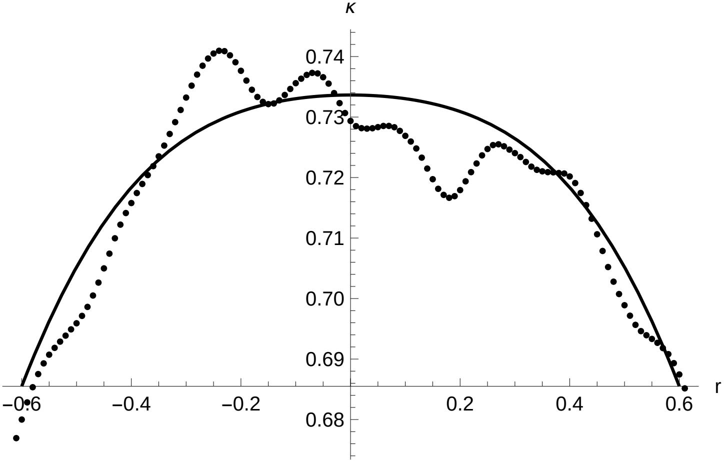


Figure: Estimating the nondimensional curvature from a data of 100 sampled points with a noise level $\delta = 10^{-3}$.

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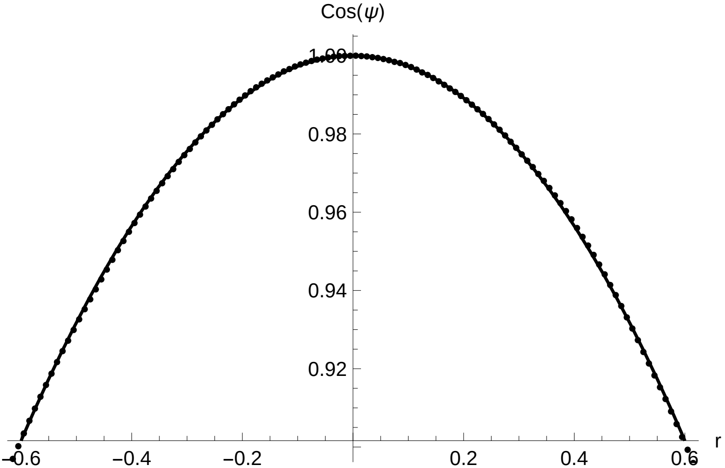

Figure: Estimating $\cos\psi$ from a data of 100 sampled points with a noise level $\delta = 10^{-3}$.

Future work and ideas

- Devise a systematic method of determining smoothing parameters.
- Consider the non-axisymmetric cornea (**essential**).
- Estimate the principal curvatures.
- Use least-squares to find α and β.
- Move to more **complex models**.

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THANK YOU!