Certified Reduced order Large Eddy Simulation turbulence models

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Sketch

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2. Smagorinsky turbulence model.
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Motivation: Energy analysis of buildings

- Official codes for energy analysis of buildings fail to accurately model the air-wall heat exchange.
- Specifically addressed numerical models obtain a large error decrease (1/3).
- There is a need for fast solvers to couple with these codes.

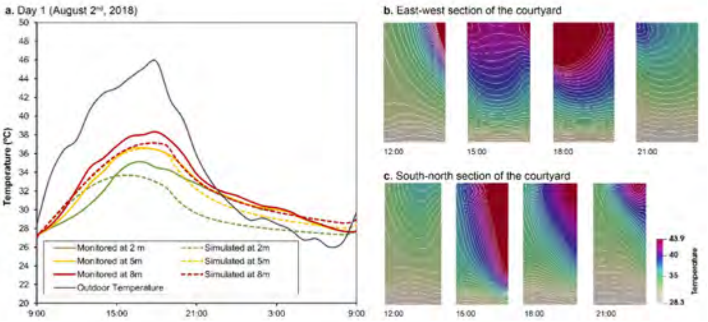

Figure: Numerical modelling of thermal behaviour of courtyard.

Smagorinsky turbulence model

We are interested in the (very) fast solution of the parametric Smagorinsky turbulence model:

$$\left\{ \begin{aligned} \partial_t \mathbf{u} - \nabla \cdot ((\nu + \nu_S(\mathbf{u}))\,\nabla \mathbf{u}) + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad &\text{in } Q_T, \\ \nabla \cdot \mathbf{u} = 0 \quad &\text{in } Q_T, \\ + \text{ initial and boundary conditions}, & \end{aligned} \right. \tag{1}$$

where $\nu_s(\mathbf{u}) = \sum_{K \in \mathcal{T}_h} (C_S h_K)^2 |\nabla \mathbf{u}_{|K}| \mathcal{X}_K$ is the eddy diffusion.

- Oriented to thermal confort in architectural design.

- Physical parameters: Reynolds number, Rayleigh number (buoyant flows, thermal flows).
- Geometrical parameters: Dimensions of building spaces.

Reduced Basis problem - steady case

- The Reduced Basis problem is defined by Galerkin projection as

$$\begin{cases} \text{Find } U_N(\boldsymbol{\mu}) = (\mathbf{u}_N(\boldsymbol{\mu}), p_N(\boldsymbol{\mu})) \in X_N \text{ such that} \\ S(U_N(\boldsymbol{\mu}), V_N; \boldsymbol{\mu}) = F(V_N; \boldsymbol{\mu}) \qquad \forall V_N \in X_N \end{cases} \tag{2}$$

where S is the Smagorinsky operator,

$$X_N = Span\{\xi_1, \cdots, \xi_{2N-1}\} \times Span\{\psi_1, \cdots, \psi_N\} : \text{ Reduced space}$$

The solution $U_N(\boldsymbol{\mu})$ can be expressed as

$$\mathbf{u}_N(\boldsymbol{\mu}) = \sum_{k=0}^{2N-1} u_k(\boldsymbol{\mu})\xi_k, \quad p_N(\boldsymbol{\mu}) = \sum_{k=0}^{N-1} p_k(\boldsymbol{\mu})\psi_k$$

- The discrete problem is constructed from parameter-independent matrices and tensors constructed off-line.
- The pair (velocity, pressure) spaces satisfies the inf-sup condition.
- Another possibility is pressure stabilization.

Greedy Algorithm

- The reduced space is constructed by a Greedy Algorithm:

1. Initialization
   - Choose a (rich enough) discrete set of parameters $\mathcal{D}_{train}$.
   - Randomly choose $\boldsymbol{\mu}_1 \in \mathcal{D}_{train}$ and set $X_1 = U_N(\boldsymbol{\mu}_1)$.
2. Enrichment. Known X_{N-1}, Compute

$$\boldsymbol{\mu}_N = \text{argmax}_{\boldsymbol{\mu} \in \mathcal{D}_{train}} \|\mathbf{u}_N(\boldsymbol{\mu}) - \mathbf{u}_{trust}(\boldsymbol{\mu})\|_X$$

and set

$$X_N = Span\{X_{N-1}, \mathbf{u}_N(\boldsymbol{\mu}_N)\}.$$

For evolution problems a further reduction of the discrete space by POD is needed.

- The Greedy Algorithm is oriented to minimize the distance in $L^\infty(\mathcal{D}, X)$ between the reduced and the trust solutions.

A posteriori error estimation: general framework

• In practice the error $\|\mathbf{u}_N(\boldsymbol{\mu}) - \mathbf{u}_{trust}(\boldsymbol{\mu})\|_X$ should be approximated by an posteriori error bound, $\Delta_N(\boldsymbol{\mu})$.

• The Brezzi-Rappaz-Raviart Theory for approximation of regular branches of non-linear variational problems is used.

• The tangent operator must be an isomorphism at each parameter of the branch.

• The a posteriori estimation holds if the tangent operator is locally Lipschitz-continuous.

A posteriori error estimation: Smagorinsky model

• The Smagorinsky operator is smoother than the Navier-Stokes one, due to the eddy viscosity term.

• The following estimates for Euler + stable Finite Element discretisation hold: For all $U_h = (u_h, p_h)$, $V_h = (v_h, q_h) \in X_h$,

$$\|\partial S(U_h, \boldsymbol{\mu}) - \partial S(V_h, \boldsymbol{\mu})\|_{\mathcal{L}(X,X')} \leq \rho_T(\boldsymbol{\mu}) \, \|U_h - V_h\|_X^{\alpha},$$

where

$$X = L^2(0,T; H_0^1(\Omega)^3) \times L_0^2(0,T;\Omega)$$

endowed with the Hilbertian norm

$$\|(u,p)\|_X = \left(\|\partial_t u\|^2_{L^2(L^2)} + \|u\|^2_{L^2(H^1)} + \|p\|^2_{L^2(L^2)} \right)^{1/2};$$

$$\alpha = \begin{cases} 2/3 & \text{for the evolution problem,} \\ 1 & \text{for the steady problem.} \end{cases}$$

A posteriori error estimation: Smagorinsky model

In the evolution problem, this holds thanks to the enhanced estimates

Theorem

Assume that $\Delta t \leq C\, h^{5/3}$. Then the solution (u_h, p_h) of the $(\mathrm{P}_2 - \mathrm{P}_1)$-Finite Element + semi-implicit time Euler discretization of the Smagorinsky model satisfies

$$\|\partial_t u_h\|_{L^2(L^2)} + \|\nabla u\|_{L^\infty(L^3)} + \|p_h\|_{L^2(L^2)} \leq C(\nu, h_{min}, \|f\|_{L^2(L^2)})$$

• The condition $\Delta t \leq C\, h^{5/3}$ is not very restrictive as in practice the grid size h is determined in such a way that a part of the inertial spectrum is resolved.

A posteriori error estimation: Smagorinsky model

- This allows to construct an error estimator

$$\Delta_N(\boldsymbol{\mu}) = \Delta_N(\boldsymbol{\mu})(\rho_T(\boldsymbol{\mu}), \varepsilon(\boldsymbol{\mu}), \beta(\boldsymbol{\mu}))$$

in terms of

- The constant $\rho_T(\boldsymbol{\mu})$ appearing in the Lipschitz or Hölder estimate for the tangent operator.
- The dual norm $\varepsilon(\boldsymbol{\mu})$ of the residual $\mathcal{R}(U_N) = A(U_N, \boldsymbol{\mu}) - F$.
- The coercivity constant $\beta(\boldsymbol{\mu})$ of the tangent operator.

- The estimator $\Delta_N(\boldsymbol{\mu})$ (solution of an algebraic equation) can be computed whenever $\varepsilon(\boldsymbol{\mu})$ is small enough.

Thermal comfort optimisation of peristyles

• Purpose: To optimise the geometrical design peristyles to reach the best thermal comfort in hot climates.

• Model: Steady Smagorinsky + Heat conservation equations, forced convection.

• The equations are transformed by a change of variables from a reference domain. This makes explicit the dependence of the operator with respect to the parameters.

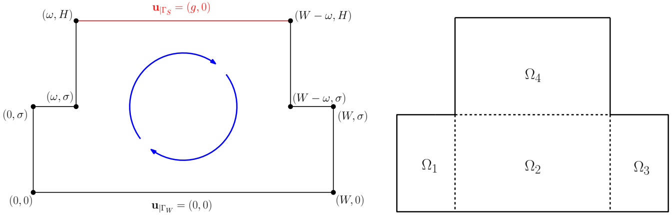

Figure: Left: Geometrical setting for targeted cloister. Right: Reference domain.

Construction of Reduced Space history

- Error estimator in terms of number of basis functions.

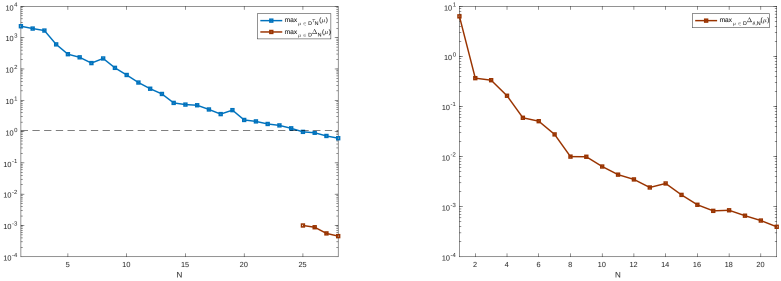

Figure: Left: Error estimator for velocity. Right: Error estimator for temperature.

- The eddy viscosity is approximated by an Empirical Interpolation technique.

Trust vs reduced solution.

- $Re = 3.100$. $T_{top} = 24^o C$, $T_{bottom} = 22^o C$. Adiabatic conditions on solid walls.

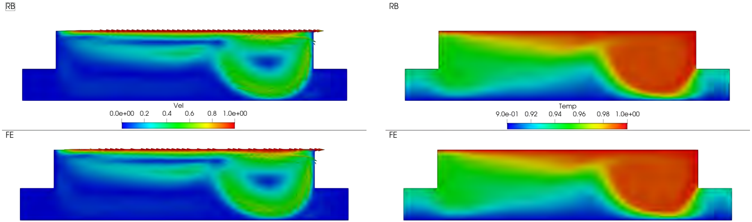

Figure: Comparison of reduced-trust velocity (left) and temperature (right).

Numerical performance of Reduced Basis method

- Errors and computational speeds-up for three cases not included in the training set $\mathcal{D}_{train}$.

  - Case 1: $\omega = 2.891$, $\sigma = 2.734$
  - Case 2: $\omega = 2.649$, $\sigma = 2.65$
  - Case 3: $\omega = 2.469$, $\sigma = 2.923$

Data	Case 1	Case 2	Case 3
$\|\|U_h - U_N\|\|_T$	$5.93 \cdot 10^{-6}$	$3.73 \cdot 10^{-6}$	$7.28 \cdot 10^{-6}$
Δ_N	$1.21 \cdot 10^{-4}$	$1.07 \cdot 10^{-4}$	$1.89 \cdot 10^{-4}$
$\|\|\theta_h - \theta_N\|\|_{L^2}$	$1.56 \cdot 10^{-5}$	$5.49 \cdot 10^{-5}$	$4.62 \cdot 10^{-5}$
$\Delta_{\theta,N}$	$3.61 \cdot 10^{-5}$	$1.49 \cdot 10^{-4}$	$1.11 \cdot 10^{-4}$
speedup	133	152	141

Thermal comfort optimization

• Purpose: To optimize the peristilyum geometry to get a temperature at bottom part as close as possible to the comfort temperature ($T_c = 24^o C$):

• Problem: Set $\mathcal{D} = [2,4] \times [2.5,3]$ (lengths in meters). Obtain

$$\underset{(\omega,\sigma)\in\mathcal{D}}{\operatorname{argmin}} J(\omega,\sigma), \text{ with } J(\omega,\sigma) = \frac{\| T(\omega,\sigma) - T_c \|_{L^2(\Omega_{Bottom})}}{\sqrt{|\Omega_{Bottom}|}}.$$

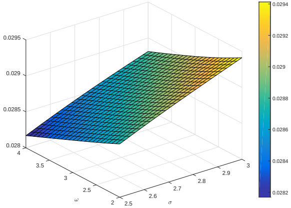

Figure: Thermal comfort functional J.

• The minimum is at $\omega = \omega_{max}$, $\sigma = \sigma_{min}$ (maximum width and minimum height of corridor).

Concluding remarks and future work

- The regularity of Smagorinsky operator allows to construct a posteriori-error estimators for RB models.
- Need of increasing the efficiency of estimators. In progress an estimator based upon the Kolmogorov theory of equilibrium turbulence.

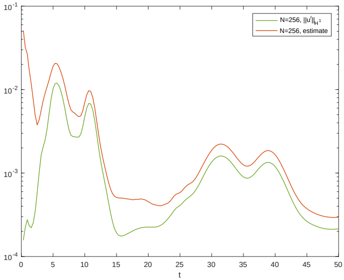

Figure: Comparison between error and estimate based upon Kolmogorov turbulence theory.

- Mixed data-driven/physics-based turbulence models in view. Modelling of effect of sub-grid scales by data-driven ROM techniques.