Lebesgue estimates for the Thresholding Greedy Algorithm

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New Bridges between Mathematics and Data Science

We know that analog images need to be digitized in order to be stored and manipulated by our computers. In fact, since an analog images could be interpreted like a function $f(x,y)$ supported in $[0,1]^2$, we can produce a digital version of $f(x,y)$ as follows:

- We can codify the image as a sequence of 2^{2m} coefficients $(p_{\mathbf{k}})$:

$$p_{\mathbf{k}} = \frac{1}{|I_{m,\mathbf{k}}|}\int\int_{I_{m,\mathbf{k}}} f(x,y)dxdy,\ 0 \leq k_1, k_2 < 2^m,$$

  where $I_{m,\mathbf{k}}$ denotes the dyadic square $\left[\frac{k_1}{2^m}, \frac{k_1+1}{2^m}\right] \times \left[\frac{k_2}{2^m}, \frac{k_2+1}{2^m}\right]$.

- $f(x,y) = (p_{\mathbf{k}})_{\mathbf{k}}$.

- For the mathematical model, this process may be reversed. Given a sequence $(p_{\mathbf{k}})_{\mathbf{k}}$, the *observed image* $f^o(x,y)$ is

$$f^o(x,y) = \sum_k p_{\mathbf{k}} \phi_{m,\mathbf{k}}(x,y),$$

where $\phi_{m,\mathbf{k}}(x) = \phi(2^m x - \mathbf{k})$ and ϕ is the characteristic function in the square or a spline or wavelet.

The problem consists in representing $f^o(x,y)$ with much less than 2^{2m} coefficients without loosing the visual resemblance with the original image.

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$$f = \sum_{n=1}^{\infty} a_n e_n.$$

- $\mathbf{1}_{\varepsilon A} = \sum_{n \in A} \varepsilon_n e_n$, $\varepsilon_n \in \{\pm 1\}$, "the indicator sum on A with signs".

  If $\varepsilon \equiv 1$, we use $\mathbf{1}_A$.

Thresholding greedy algorithm

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The m-**th greedy sum** of f is the sum $\mathcal{G}_m(f) = \sum_{j \in A(f)} a_j e_j$,

$$\min_{j \in A(f)} |a_j| \geq \max_{j \notin A(f)} |a_j|.$$

The set $A(f)$ is called **greedy set** and the collection $\{\mathcal{G}_m\}_m$ is the **Greedy Algorithm**.

Our goal

How good is $\|f - \mathcal{G}_m(f)\|$ vs $\sigma_m(f)$, where

$$\sigma_m(f) := \inf\{\|f - \sum_{n \in A} c_n e_n\| \ : \ |A| = m, \ c_n' s \text{ scalars}\}.$$

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Goal: find smallest $\mathbf{L}_m = \mathbf{L}_m(\mathbb{X}, \mathcal{B})$ such that

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2. If $\sup_m \mathbf{L}_m = \infty$, we want to study the growth of the constant $\mathbf{L}_m$.

Greedy-type bases

Section 1

Quasi-greedy bases

Definition

A basis $\mathcal{B}$ of a Banach space $\mathbb{X}$ is **quasi-greedy** if there exists a constant C such that for any $f \in \mathbb{X}$ and $m \in \mathbb{N}$ we have

$$\|f - \mathcal{G}_m(f)\| \leq C\|f\|.$$

Theorem (Wojtasczyk; 2000)

A basis is quasi-greedy if and only if

$$\lim_{m\to\infty} \|f - \mathcal{G}_m(f)\| = 0, \ \forall f \in \mathbb{X}.$$

P. Wojtaszczyk, *Greedy algorithm for general biorthogonal systems*, J.Approx.Theory **107** (2000), no.2, 293-314.

Greedy Bases

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A basis $\mathcal{B}$ of a Banach space $\mathbb{X}$ is **greedy** if there exists an absolute constant $C \geq 1$ such that

$$\sigma_m(f) \leq \|f - \mathcal{G}_m(f)\| \leq C\sigma_m(f), \;\; \forall m \in \mathbb{N}, \;\; \forall f \in \mathbb{X}.$$

Democratic and Unconditional basis

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We say that a basis $\mathcal{B}$ of a Banach space $\mathbb{X}$ is **democratic** if there exists $C_d \geq 1$ such that for any finite sets A, B of the same cardinality, we have that

$$C_d^{-1}\|\mathbf{1}_B\| \leq \|\mathbf{1}_A\| \leq C_d\|\mathbf{1}_B\|.$$

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$\mathbb{X} = \ell^p, 1 \leq p < \infty$ and $\mathcal{B}$ the canonical basis. Let A, B with $|A| = |B|$, then

$$\|\mathbf{1}_A\|_p = |A|^{1/p} = |B|^{1/p} = \|\mathbf{1}_B\|_p.$$

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Definition

We say that a basis $\mathcal{B}$ of a Banach space $\mathbb{X}$ is **suppression unconditional** if there exists $K \geq 1$ such that $\forall f \in \mathbb{X}$ and $\forall A \subset \mathbb{N}$,

$$\|P_A(f)\| \leq K_s\|f\|, \quad P_A\left(\sum_{j=1}^{\infty} a_i e_i\right) = \sum_{j\in A} a_i e_i.$$

Theorem (Temlyakov, Konyagin; 1999)

A basis $\mathcal{B}$ is greedy if and only if $\mathcal{B}$ is democratic and unconditional.

S.V.Konyagin, V.N.Temlyakov, *A remark on greedy approximation in Banach spaces*, East J. Approx. **5** (1999), 365-379.

Examples

1. Let $\mathbb{X} = \mathbb{H}$ be a Hilbert space and $\mathcal{B} = (e_n)_n$ an orthonormal basis. Then $\mathcal{B}$ is greedy with constant $C_g = 1$,
$$\|f - \mathcal{G}_m(f)\| = \sigma_m(f)\ \forall f \in \mathbb{H},\ \forall m \in \mathbb{N}$$

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3. The Haar basis is a greedy basis in $L_p[0,1]$ with $1 < p < \infty$ and the constant is 1 if $p = 2$. (Temlyakov)

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4. The trigonometric system is not greedy in $L_p(\mathbb{T})$ with $p \neq 2$. (Temlyakov)

   V.N.Temlyakov, *Greedy Algorithm and m-Term Trigonometric Approximation*, Constr. Approx. **14** (1998), no. 4, 569-587.

Lebesgue-type inequalities for the greedy algorithm in general bases

Section 2

Lebesgue-type constant

To quantify the efficiency of greedy approximation one defines, for each $m = 1, 2, ...$ the smallest number $\mathbf{L}_m$ such that

$$\|f - \mathcal{G}_m(f)\| \leq \mathbf{L}_m \sigma_m(f), \ \ \forall f \in \mathbb{X}.$$

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Historical remark: Lebesgue prove in 1909 the following: for any 2π-periodic and continous function f we have

$$\frac{\|f - S_m(f)\|_\infty}{E_m(f)_\infty} \approx \ln(m),$$

where $S_m(f)$ is the m-th partial sum of the Fourier series of f and $E_m(f)_\infty$ is the error of the best approximation of f by trigonometric polynomials of order m in the uniform norm $\|\cdot\|_\infty$.

The trigonometric system

The trigonometric system $\mathcal{T} = \{e^{2\pi ix}\}_{k\in\mathbb{Z}}$ in $L^p(\mathbb{T})$, with $1 < p < \infty$, is not greedy with $p \neq 2$. Moreover, Temlyakov showed that

$$\mathbf{L}_m \approx m^{|\frac{1}{2}-\frac{1}{p}|}.$$

V.N.Temlyakov, *Greedy Algorithm and m-Term Trigonometric Approximation*, Constr. Approx. **14** (1998), no. 4, 569-587.

Constants

- **Conditionality constant**:

$$k_m = \sup_{|A|\leq m} \|P_A\|, \;\; k_m^c = \sup_{|A|\leq m} \|I - P_A\|.$$

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- **Democracy constants**:

$$\mu_m = \sup_{|A|=|B|\le m} \frac{\|\mathbf{1}_A\|}{\|\mathbf{1}_B\|} \text{ and } \tilde{\mu}_m = \sup_{\substack{|A|=|B|\le m \\ \varepsilon,\varepsilon'\in\{\pm 1\}}} \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\varepsilon' B}\|}.$$

Theorem (P. Oswald)

If $\mathcal{K} = \sup_{m,n} \|e_m\| \|e_n^*\|$, then for all $m \geq 1$ we have

$$1 \leq \mathbf{L}_m \leq 1 + 3m\mathcal{K}.$$

The Haar system in $L^1([0,1])$ satisfies $\mathbf{L}_m = 1 + 3m$.

P. Oswald, *Greedy algorithms and best m-term approximation with respect to biorthogonal systems*, J. Fourier Anal. Appl. **7** (2001), 325-341.

Some results in the context of quasi greedy bases

1) $\mathbf{L}_m \leq 1 + 2k_m + 8C_{qg}^3 \mu_m k_m$.

V.N. Temlyakov, M. Yang, P. Ye, *Lebesgue-type inequalities for greedy approximation with respect to quasi-greedy bases*, East J. Approx **17** (2011), 127-138.

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In c_{00} we define the norm of a sequence $(a_n)_{n=1}^{\infty}$ by the formula

$$\|(a_n)_n\| = \max\left\{\|(a_n)_n\|_2, \sup_N \left|\sum_{n=1}^{N} \frac{a_n}{\sqrt{n}}\right|\right\}.$$

Let $\mathbb{X}$ be the completion of c_{00} in c_0 under this norm and let $\mathcal{B} = (e_n)_n$ the canonical basis in $\mathbb{X}$. This basis is quasi-greedy and democratic.

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Let $\mathbb{X}$ be the completion of c_{00} in c_0 under this norm and let $\mathcal{B} = (e_n)_n$ the canonical basis in $\mathbb{X}$. This basis is quasi-greedy and democratic. Now, if we define $\mathbb{Y} = \mathbb{X} \oplus c_0$, the canonical basis is quasi-greedy and not democratic.

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2) $\max\{k_m, \frac{\mu_m}{3C_{qg}}\} \lesssim \mathbf{L}_m \leq 1 + 2k_m + 8C_{qg}^4 \mu_m$.

G. Garrigós, E. Hernández, T. Oikgberg, *Lebesgue-type inequalities for quasi-greedy bases*, Constr. Approx. **38** (2013), 447-470.

The trigonometric system is not quasi-greedy!!

New result in the context of superdemocratic bases

Theorem (Berná, Blasco, Garrigós; 2017)

For all $m \geq 1$ we have

$$\max\left\{k_m, \mu_m, \frac{\tilde{\boldsymbol{\mu}}_{\boldsymbol{m}}}{\boldsymbol{2\kappa}}\right\} \lesssim \mathbf{L}_m \lesssim \boldsymbol{k_m + k_m \tilde{\mu}_m}$$

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$\mathbb{S}_1 \hookrightarrow \mathbb{X}$

Discrete weighted Lorentz spaces: let $\eta = \{\eta(j)\}_j$ a positive non-decreasing weight,

$$\ell^1_\eta = \{s \in c_0 : \|s\|_{\ell^1_\eta} := \sum_{j=1}^{\infty} s^*_j \frac{\eta(j)}{j} < \infty\}.$$

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Let $\mathcal{B}$ a basis in $\mathbb{X}$ and η a non-decreasing positive weight. The following are equivalent:

1. $\|1_{\varepsilon A}\| \leq \eta(|A|)$ for all finite $A \subset \mathbb{N}$ and all $|\varepsilon| = 1$.
2. $\ell^1_{\hat{\eta}} \overset{\mathcal{B},\,1}{\hookrightarrow} \mathbb{X}$, where $\hat{\eta}(j) = j\Delta\eta(j) = j(\eta(j) - \eta(j-1))$.

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- If η is doubling, ℓ^1_η and $\ell^1_{\hat{\eta}}$ are quasi-normed.

$\mathbb{X} \hookrightarrow \mathbb{S}_2$

Discrete weighted Marcinkiewicz spaces: let $\eta = \{\eta(j)\}_j$ a positive weight,

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We can take $\eta(n) := \varphi^*(n) = \sup_{\substack{|\varepsilon|=1 \\ |A| \leq n}} \|\mathbf{1}^*_{\varepsilon A}\|_*$.

- For every positive weight η, $m(\eta)$ is a normed space.
- If η is doubling and $\inf_k \frac{\eta(k)}{k} = 0$, then $(\ell^1_{\hat{\eta}})^* = m(\eta')$.

New theorem for general bases

Theorem (Berná, Blasco, Garrigós, Hernández, Oikhberg; 2017)

Let $\mathcal{B}$ a basis in $\mathbb{X}$. Then the following hold:

$$k_m \leq \overline{T}_m(\varphi,\varphi^*),\ \mathbf{L}_m \leq 1+3\overline{T}_m(\varphi,\varphi^*),$$

where

$$T_m(\varphi,\varphi^*) := \sum_{j=1}^{m} \frac{\varphi(j)}{j}\Delta\varphi^*(j),\ \Delta\varphi(j) = \varphi(j)-\varphi(j-1),$$

$$\overline{T}_m(\varphi,\varphi^*) := \min\{T_m(\varphi,\varphi^*), T_m(\varphi^*,\varphi)\}.$$

Finally, these estimates are best possible, in the sense that there exist $\mathbb{X}$ and $\{e_j, e_j^*\}_{j=1}^{\infty}$ for which all the equalities hold.

The trigonometric system

The trigonometric system $\mathcal{T} = \{e^{2\pi ix}\}_{k\in\mathbb{Z}}$ in $L^p(\mathbb{T})$, $1 < p < \infty$.

- For $1 < p < 2$, $\varphi(m) \leq m^{1/2}$, and $\varphi^*(m) \leq m^{1/p}$.

$$\overline{T}_m(\varphi, \varphi^*) \leq \sum_{j=1}^{m} j^{-1/2} j^{1/p-1} \leq c_p m^{1/p-1/2}.$$

- For $2 < p < \infty$, $\varphi(m) \leq m^{1/p'}$, and $\varphi^*(m) \leq m^{1/2}$.

$$\overline{T}_m(\varphi, \varphi^*) \leq \sum_{j=1}^{m} j^{1/p'-1} j^{-1/2} \leq c_p m^{1/p'-1/2} = c_p m^{1/2-1/p}.$$

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Drawback: we cannot recover the trivial case $p = 2$.

Latest results

For each $m = 1, 2, ...$, $\mathbf{C}_m$ is the smallest constant C such that

$$\min_{n \in A} |a_n| \varphi(|A|) \leq C\|f\|, \forall f \in \mathbb{X},$$

where $f = \sum a_n e_n$.

If $\sup_m \mathbf{C}_m < +\infty$, we say that $\mathcal{B}$ has the Property (C*).

- Every unconditional (or quasi-greedy) basis has the Property (C*).
- We have proved the existence of some examples where $\mathcal{B}$ has the Property (C*) and the basis is not quasi-greedy.

Theorem (AAB, 2021)

For each $m = 1, 2, ...$, there are two constant C_1 and C_2 such that

$$C_1 \max\{k_m,\, \mathbf{C}_m\} \leq \mathbf{L}_m \leq C_2 \max\{k_m,\, \mathbf{C}_m\}.$$

First optimal result for every basis in a Banach (or quasi-Banach) space

F. Albiac, J. L. Ansorena, P. M. Berná, *New parameters and Lebesgue-type estimates in greedy approximation*, Submitted (2021).

New project: subspace learning by greedy algorithms

Section 3

Tensor-based methods are receiving a growing interest in scientific computing for the numerical solution of problems defined in high dimensional tensor product spaces, such as partial differential equations arising from stochastic calculus or quantum mechanics.

These methods consist in approximation the solution $\mathbf{u} \in V$ of a problem (V is a tensor space generated by d vector spaces V_j), using representations of the form

$$\mathbf{u} \approx \mathbf{u}_m = \sum_{i=1}^{m} w_i^{(1)} \otimes \cdots \otimes w_i^{(d)},\ w_i^{(j)} \in V_j,$$

where $\otimes$ represents the Kronecker product.

A particular family of methods, called Proper Generalized Decomposition (PGD) methods, have been introduced for the direct construction of representations of type(1).

A. Falcó, A. Nouy, *Proper generalized decomposition for nonlinear convex problems in tensor Banach spaces*, Numer. Math. (2012) 121:503–530.

Consider $(\mathbb{X}, \|\cdot\|)$ be a Banach space, $\mathbb{X}^*$ is the dual space and

$$\langle \cdot, \cdot \rangle : \mathbb{X}^* \times \mathbb{X} \to \mathbb{R}$$

the duality pairing.
We consider the following optimization problem (P):

$$\min_{x \in \mathbb{X}} J(x) = J(\overline{x}),$$

where J is a given functional verifying some properties:

- J is Fréchet differentiable.
- There is $\alpha > 0$ and $s > 1$ such that

$$\langle J'(x) - J'(y), x - y \rangle = \alpha \|x - y\|^s.$$

Remark: under these assumptions, J is strictly convex.

Progressive proper generalized decompositions in Banach spaces

Definition

We say that $\mathcal{D} \subset \mathbb{X}$ is a **dictionary** if

- $\mathcal{D}$ is weakly closed in $\mathbb{X}$.
- $\forall d \in \mathcal{D}$, $\lambda\, d \in \mathcal{D}$, $\forall \lambda \in \mathbb{R}$.
- $\overline{\text{span}}(\mathcal{D}) = \mathbb{X}$.

Theorem

If $\overline{x} \in \mathbb{X}$ is an element such that

$$J(\overline{x}) = \min_{z \in \mathcal{D}} J(\overline{x} + z),$$

then $\overline{x}$ solves the problem (P).

Progressive proper generalized decompositions in Banach spaces

Let $\mathbf{u}_0 = 0$ and take $m \geq 1$.

- Find $\hat{\mathbf{z}}_m$ such that

$$J(\mathbf{u}_{m-1} + \hat{\mathbf{z}}_m) = \min_{z \in \mathcal{D}} J(\mathbf{u}_{m-1} + z). \quad (*)$$

- Next before to update m to $m+1$, we can have two possible possibilities:
  1. Take $\mathbf{z}_m = \hat{\mathbf{z}}_m$ and $\mathbf{u}_m = \mathbf{u}_{m-1} + \mathbf{z}_m$, update to m to $m+1$ and go to $(*)$.
  2. Take another strategy such that we take $\tilde{z}_m$ so that

$$J(\mathbf{u}_{m-1} + \tilde{z}_m) \leq J(\mathbf{u}_{m-1} + \hat{\mathbf{z}}_m).$$

     In that case, $z_m = \tilde{z}_m$, $\mathbf{u}_m = \mathbf{u}_{m-1} + \mathbf{z}_m$ and update m to $m+1$ and go to $(*)$.

Progressive proper generalized decompositions in Banach spaces

Proposition

For each $m \geq 1$,

$$J(\mathbf{u}_{m-1}) \leq J(\mathbf{u}_m).$$

Moreover, if $J(\mathbf{u}_m) = J(\mathbf{u}_{m-1})$, then $\mathbf{u}_{m-1}$ is the solution of (P).

Theorem

Let $\mathbf{u} \in \mathbb{X}$ satisfying $J(\mathbf{u}) = \min_{x \in \mathbb{X}} J(x)$. Then, every greedy progressive PGD $(\mathbf{u}_m)_{m \geq 1}$ over $\mathcal{D}$ converges in $\mathbb{X}$ to $\mathbf{u}$.

Progressive proper generalized decompositions in Banach spaces

One of the possible strategies in the Step 2 could be the following: define $\mathbf{U} = \text{span}\{\mathbf{z}_1, \cdots, \mathbf{z}_m\}$, with $\dim(\mathbf{U}) = m$.
In that step, we compute $\tilde{\mathbf{z}}_m$ as

$$J(\mathbf{u}_{m-1} + \tilde{\mathbf{z}}_m) = \min_{z \in \mathbf{U}} J(\mathbf{u}_{m-1} + z).$$

Remark: in the context of Greedy Algorithms for computing best approximations, an update of the last form by using an orthonormal basis of $\mathbf{U}$ corresponds to an orthogonal Greedy Algorithm.

Thank you for your attention!