POD stabilized methods for incompressible flows: error analysis and computational results

Samuele Rubino[a]
(joint work with *Julia Novo*[b])

[a]Universidad de Sevilla
[b]Universidad Autónoma de Madrid

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Outline

1. Mathematical formulation of a stabilized **POD-ROM**: approximation of an IBVP for the *Navier–Stokes equations*.
   - POD-ROM methodology + Local Projection Stabilization (**LPS-ROM**).
2. **Velocity-pressure LPS-ROM**:
   (I) Computation of relevant quantities (e.g., **drag/lift** forces).
   (II) Circumvent the **inf-sup** condition.
   (III) NO **divergence-free** velocity modes required.
3. Numerical analysis of the **LPS-ROM** in the context of stabilized **FEM** for the *Navier–Stokes equations*.
   - Stability analysis + Error estimates (indep. of ν^{-1}).
4. Comparison to a POD-ROM based on *supremizers* (**grad-div-ROM**).
5. Application to numerical simulation of:
   - *2D Unsteady flow past a circular obstacle* ($Re = 10^2$).

Mathematical formulation

- $\Omega \subset \mathbb{R}^d$ = bounded domain ($d = 2, 3$).
- $\Gamma = \partial\Omega$ = Lipschitz boundary.

Incompressible NSE with homogeneous Dirichlet BC:

Find $\mathbf{u} : \Omega \times (0, T) \longrightarrow \mathbb{R}^d$ *and* $p : \Omega \times (0, T) \longrightarrow \mathbb{R}$ *s.t.:*

$$\left\{ \begin{array}{rcl} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p & = & \mathbf{f}, \\ \nabla \cdot \mathbf{u} & = & 0, \\ \mathbf{u} & = & \mathbf{0} \text{ on } \Gamma, \\ \mathbf{u}(\mathbf{x}, 0) & = & \mathbf{u}_0(\mathbf{x}). \end{array} \right. \tag{1}$$

Variational formulation

- $\mathbf{X} = \left[H_0^1(\Omega)\right]^d$, $Q = L_0^2(\Omega)$ (Velocity, pressure spaces).

Variational Formulation:

Find $\mathbf{u} : (0, T) \longrightarrow \mathbf{X}$ *and* $p : (0, T) \longrightarrow Q$ *s.t.:*

$$\left\{ \begin{array}{rcll} \dfrac{d}{dt}(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) + \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) & = & \langle \mathbf{f}, \mathbf{v} \rangle & \forall \mathbf{v} \in \mathbf{X}, \\ (\nabla \cdot \mathbf{u}, q) & = & 0 & \forall q \in Q, \\ \mathbf{u}(0) & = & \mathbf{u}_0, & \end{array} \right. \tag{2}$$

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \frac{1}{2} \left[(\mathbf{u} \cdot \nabla \, \mathbf{v}, \mathbf{w}) - (\mathbf{u} \cdot \nabla \, \mathbf{w}, \mathbf{v}) \right], \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbf{X}. \tag{3}$$

Full Order Model (LPS-FOM)

FE approximation:

- $\{\mathcal{T}_h\}_{h>0}$: Affine-equivalent, conforming and regular.
- $\mathbf{X}_h \subset \mathbf{X}$, $Q_h \subset Q$: Suitable **FE** velocity, pressure spaces.

Galerkin FEM: *Find* $(\mathbf{u}_h, p_h) : (0,T) \longrightarrow \mathbf{X}_h \times Q_h$ *s.t.*

$$\begin{cases} \dfrac{d}{dt}(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) = \langle \mathbf{f}, \mathbf{v}_h \rangle & \forall \mathbf{v}_h \in \mathbf{X}_h, \\ (\nabla \cdot \mathbf{u}_h, q_h) = 0 & \forall q_h \in Q_h, \\ \mathbf{u}_h(0) = \mathbf{u}_{0h}. \end{cases} \quad (4)$$

- Circumvent the **inf-sup** condition and use **Equal Order-FE**.
- **Offline LPS by interpolation procedure**: offline control on the high frequencies component of the velocity/pressure gradient.

J. de Frutos et al., *Error analysis of non inf-sup stable discretizations of the time-dependent NSE with LPS*, **IMAJNA**, (2019).

LPS by interpolation FEM

Equal Order-FE velocity, pressure spaces:

$$\left\{ \begin{array}{l} Y_h^l = \{v_h \in C^0(\overline{\Omega}) : v_h|_K \in \mathbb{P}_l(K), \forall K \in \mathcal{T}_h\}, \quad l \geq 2, \\ \mathbf{X}_h^l = [Y_h^l]^d \cap \mathbf{X}, \quad Q_h^l = Y_h^l \cap Q. \end{array} \right. \tag{5}$$

Definition [τ-scalar product and norm]

$$(\cdot,\cdot)_\tau : \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega) \to \mathbb{R}, \quad (\mathbf{v},\mathbf{w})_\tau = \sum_{K\in\mathcal{T}_h} \tau_K (\mathbf{v},\mathbf{w})_K,$$

$$\|\mathbf{v}\|_\tau = (\mathbf{v},\mathbf{v})_\tau^{1/2}.$$

LPS by interpolation FEM

LPS by interpolation method: *Find* $(\mathbf{u}_h, p_h) : (0, T) \longrightarrow \mathbf{X}_h^l \times Q_h^l$ *s.t.*

$$\left\{ \begin{array}{rcll} \frac{d}{dt}(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) & & & \\ +(\sigma_h^*(\nabla \mathbf{u}_h), \sigma_h^*(\nabla \mathbf{v}_h))_{\tau_\nu} - (p_h, \nabla \cdot \mathbf{v}_h) & = & \langle \mathbf{f}, \mathbf{v}_h \rangle & \forall \mathbf{v}_h \in \mathbf{X}_h^l, \\ (\nabla \cdot \mathbf{u}_h, q_h) + (\sigma_h^*(\nabla p_h), \sigma_h^*(\nabla q_h))_{\tau_p} & = & 0 & \forall q_h \in Q_h^l, \\ \mathbf{u}_h(0) & = & \mathbf{u}_{0h}. & \end{array} \right. \quad (6)$$

High-order stabilization:

- $\sigma_h^* = Id - \sigma_h : \mathbf{L}^2 \to [V_h^{l-1}]^d =$ *Continuous "buffer" space*.
- $\sigma_h =$ *Scott–Zhang-like interpolation operator*.

T. Chacón et al., *A high-order term-by-term stabilization solver for incompressible flow problems*, **IMAJNA**, (2013).

LPS by interpolation FEM

Full space-time discretization (Backward Euler time stepping):

Find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) : (0,T) \longrightarrow \mathbf{X}_h^l \times Q_h^l$ *s.t.* $\forall (\mathbf{v}_h, q_h) \in \mathbf{X}_h^l \times Q_h^l$

$$\left\{ \begin{array}{rcl} \left(\dfrac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{\Delta t}, \mathbf{v}_h\right) + b(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}, \mathbf{v}_h) + \nu(\nabla \mathbf{u}_h^{n+1}, \nabla \mathbf{v}_h) & & \\ +(\sigma_h^*(\nabla \mathbf{u}_h^{n+1}), \sigma_h^*(\nabla \mathbf{v}_h))_{\tau_\nu} - (p_h^{n+1}, \nabla \cdot \mathbf{v}_h) & = & \langle \mathbf{f}^{n+1}, \mathbf{v}_h \rangle, \\ (\nabla \cdot \mathbf{u}_h^{n+1}, q_h) + (\sigma_h^*(\nabla p_h^{n+1}), \sigma_h^*(\nabla q_h))_{\tau_p} & = & 0, \\ \mathbf{u}_h^0 & = & \mathbf{u}_{0h}. \end{array} \right. \tag{7}$$

High-order stabilization:

- $\sigma_h^* = Id - \sigma_h : \mathbf{L}^2 \rightarrow [V_h^{l-1}]^d =$ *Continuous "buffer" space*.
- $\sigma_h =$ *Scott–Zhang-like interpolation operator*.

Reduced Order Model (LPS-ROM)

"Equal Order"-POD velocity, pressure spaces: (L^2-POD basis)

- $\mathbf{U}^r = \text{span}\{\boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_r\} \subset \mathbf{X}_h^l \Rightarrow \mathbf{u}_r(\mathbf{x}, t) = \sum_{i=1}^{r} a_i(t)\boldsymbol{\varphi}_i(\mathbf{x}).$
- $\mathcal{W}^r = \text{span}\{\psi_1, \dots, \psi_r\} \subset Q_h^l \Rightarrow p_r(\mathbf{x}, t) = \sum_{i=1}^{r} b_i(t)\psi_i(\mathbf{x}).$

Local Projection Stabilization POD-ROM:

Find $(\mathbf{u}_r^{n+1}, p_r^{n+1}) : (0, T) \longrightarrow \mathbf{U}^r \times \mathcal{W}^r$ *s.t.* $\forall(\boldsymbol{\varphi}, \psi) \in \mathbf{U}^r \times \mathcal{W}^r$

$$\left\{\begin{array}{rcl} \left(\dfrac{\mathbf{u}_r^{n+1} - \mathbf{u}_r^n}{\Delta t}, \boldsymbol{\varphi}\right) + b(\mathbf{u}_r^{n+1}, \mathbf{u}_r^{n+1}, \boldsymbol{\varphi}) + \nu(\nabla \mathbf{u}_r^{n+1}, \nabla \boldsymbol{\varphi}) & & \\ +\mu(\nabla \cdot \mathbf{u}_r^{n+1}, \nabla \cdot \boldsymbol{\varphi}) + (\sigma_h^*(\nabla \mathbf{u}_r^{n+1}), \sigma_h^*(\nabla \boldsymbol{\varphi}))_{\tau_\nu} - (p_r^{n+1}, \nabla \cdot \boldsymbol{\varphi}) & = & \langle \mathbf{f}^{n+1}, \boldsymbol{\varphi}\rangle, \\ (\nabla \cdot \mathbf{u}_r^{n+1}, \psi) + (\sigma_h^*(\nabla p_r^{n+1}), \sigma_h^*(\nabla \psi))_{\tau_p} & = & 0. \end{array}\right. \tag{8}$$

Reduced Order Model (LPS-ROM)

Local Projection Stabilization POD-ROM:

Find $(\mathbf{u}_r^{n+1}, p_r^{n+1}) : (0, T) \longrightarrow \mathbf{U}^r \times \mathcal{W}^r$ *s.t.* $\forall (\varphi, \psi) \in \mathbf{U}^r \times \mathcal{W}^r$

$$\left\{ \begin{array}{rcl} \left(\dfrac{\mathbf{u}_r^{n+1} - \mathbf{u}_r^n}{\Delta t}, \varphi\right) + b(\mathbf{u}_r^{n+1}, \mathbf{u}_r^{n+1}, \varphi) + \nu(\nabla \mathbf{u}_r^{n+1}, \nabla \varphi) & & \\ +\mu(\nabla \cdot \mathbf{u}_r^{n+1}, \nabla \cdot \varphi) + (\sigma_h^*(\nabla \mathbf{u}_r^{n+1}), \sigma_h^*(\nabla \varphi))_{\tau_\nu} - (p_r^{n+1}, \nabla \cdot \varphi) & = & \langle \mathbf{f}^{n+1}, \varphi \rangle, \\ (\nabla \cdot \mathbf{u}_r^{n+1}, \psi) + (\sigma_h^*(\nabla p_r^{n+1}), \sigma_h^*(\nabla \psi))_{\tau_p} & = & 0. \end{array} \right. \tag{9}$$

- Circumvent the **inf-sup** condition.
- **Online grad-div term**: error bounds independent of ν^{-1}.
- **Online LPS terms**: online control on the high frequencies component of the velocity/pressure gradient.

J. Novo and S. Rubino, *Error analysis of POD stabilized methods for incompressible flows*, **SINUM**, (2021).

Existence and stability results

Theorem [Existence and unconditional stability]

The fully discrete **LPS-ROM** admits a solution that satisfies:

$$\begin{aligned} & \|\mathbf{u}_r^k\|_{\mathbf{L}^2}^2 + \sum_{n=0}^{N-1} \|\mathbf{u}_r^{n+1} - \mathbf{u}_r^n\|_{\mathbf{L}^2}^2 + \Delta t \sum_{n=0}^{N-1} \mu \|\nabla \cdot \mathbf{u}_r^{n+1}\|_{L^2}^2 \\ & + \Delta t \sum_{n=0}^{N-1} \left(\nu \|\nabla \mathbf{u}_r^{n+1}\|_{\mathbf{L}^2}^2 + \|\sigma_h^*(\nabla \mathbf{u}_r^{n+1})\|_{\tau_\nu}^2 + \|\sigma_h^*(\nabla p_r^{n+1})\|_{\tau_p}^2 \right) \\ \leq \quad & \|\mathbf{u}_r^0\|_{\mathbf{L}^2}^2 + \frac{4\Delta t}{\nu} \sum_{n=0}^{N-1} \|\mathbf{f}^{n+1}\|_{\mathbf{H}^{-1}}^2, \quad k = 0, \ldots, N. \end{aligned} \tag{10}$$

S. Rubino, *Numerical analysis of a projection-based stabilized POD-ROM for incompressible flows*, **SINUM**, (2020).

Existence and stability results

Remark [Unconditional pressure stability]

$$|||\cdot||| = \sup_{\varphi \in \mathbf{U}^r} \frac{(\cdot, \nabla \cdot \varphi)}{\|\nabla \varphi\|_{\mathbf{L}^2}} + \|\sigma_h^*(\nabla \cdot)\|_{\tau_p} \Rightarrow \widetilde{\mathcal{W}}^r = \textit{Induced space.} \tag{11}$$

- $\widetilde{p}_r = \mathbb{P}^0$ in time function that takes the value p_r^{n+1} on (t_n, t_{n+1}).
- $P_r(t) = \int_0^t \widetilde{p}_r(s)\, ds$.

$$\|P_r\|_{L^\infty(\widetilde{\mathcal{W}}^r)} \leq \frac{C}{\nu^2} \quad \left[\|\widetilde{p}_r\|_{L^1(\widetilde{\mathcal{W}}^r)} \leq \frac{C}{\nu^2 \sqrt{\Delta t}} \right]. \tag{12}$$

T. Chacón and R. Lewandowski, *Mathematical and Numerical Foundations of Turbulence Models and Applications*, **Birkhäuser**, (2014).

Error analysis

Theorem [Velocity error estimates]

Under proper regularity assumption and hypothesis on the stabilization parameters, the solution of the fully discrete **LPS-ROM** satisfies:

$$\begin{aligned} &\Delta t \sum_{n=0}^{N-1} \|\mathbf{u}^{n+1} - \mathbf{u}_r^{n+1}\|_{\mathbf{L}^2}^2 \\ \leq\ & C\left[\left(h^{2l+1} + \Delta t^2\right) + (d_v - r)\Delta t^{-1}\left(h^{2l+1} + \Delta t^2\right)\right] \\ +\ & C\left[\sum_{k=r+1}^{d_v} \lambda_k + (\nu + \mu + 1 + h)\|S^v\|_2 \sum_{k=r+1}^{d_v} \lambda_k + (\mu^{-1} + h\|S^p\|_2) \sum_{k=r+1}^{d_p} \gamma_k\right]. \end{aligned}$$

Remark [Pressure error estimates]

$$\|P - P_r\|_{\ell^\infty(\widetilde{\mathcal{W}}^r)} := \max_{n=1,\dots,N} |||P^n - P_r^n||| \leq C(\nu^{-1}).$$

grad-div-ROM from an inf-sup stable FOM

Full space-time discretization (grad-div-FEM):

Find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) : (0,T) \longrightarrow \mathbf{X}_h^l \times Q_h^{l-1}$ *s.t.* $\forall (\mathbf{v}_h, q_h) \in \mathbf{X}_h^l \times Q_h^{l-1}$

$$\left\{ \begin{array}{rcl} \left(\dfrac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \mathbf{v}_h\right) + b(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}, \mathbf{v}_h) + \nu(\nabla \mathbf{u}_h^{n+1}, \nabla \mathbf{v}_h) & & \\ +\mu(\nabla \cdot \mathbf{u}_h^{n+1}, \nabla \cdot \mathbf{v}_h) - (p_h^{n+1}, \nabla \cdot \mathbf{v}_h) & = & \langle \mathbf{f}^{n+1}, \mathbf{v}_h \rangle, \\ (\nabla \cdot \mathbf{u}_h^{n+1}, q_h) & = & 0. \end{array} \right.$$

Full space-time discretization (grad-div-ROM):

Find $\mathbf{u}_r^{n+1} : (0,T) \longrightarrow \mathbf{U}^r$ *s.t.* $\forall \boldsymbol{\varphi} \in \mathbf{U}^r$

$$\left\{ \begin{array}{rcl} \left(\dfrac{\mathbf{u}_r^{n+1} - \mathbf{u}_r^n}{\Delta t}, \boldsymbol{\varphi}\right) + b(\mathbf{u}_r^{n+1}, \mathbf{u}_r^{n+1}, \boldsymbol{\varphi}) + \nu(\nabla \mathbf{u}_r^{n+1}, \nabla \boldsymbol{\varphi}) & & \\ +\mu(\nabla \cdot \mathbf{u}_r^{n+1}, \nabla \cdot \boldsymbol{\varphi}) & = & \langle \mathbf{f}^{n+1}, \boldsymbol{\varphi} \rangle. \end{array} \right.$$

Reduced pressure recovery (grad-div-ROM): *supremizer* enrichment algorithm.

Error analysis

Theorem [Velocity error estimates]

The solution of the fully discrete **grad-div-ROM** satisfies:

$$\Delta t \sum_{n=0}^{N-1} \|\mathbf{u}^{n+1} - \mathbf{u}_r^{n+1}\|_{\mathbf{L}^2}^2$$

$$\leq \quad C\left[\left(h^{2l} + \Delta t^2\right) + \sum_{k=r+1}^{d_v} \lambda_k + (\nu + \mu + 1)\|S^v\|_2 \sum_{k=r+1}^{d_v} \lambda_k\right].$$

Theorem [Pressure error estimates]

$$\Delta t \sum_{n=0}^{N-1} \|p^{n+1} - p_r^{n+1}\|_{\mathbf{L}^2}^2$$

$$\leq \quad C\left[\left(h^{2l} + \Delta t^2\right) + \sum_{k=r+1}^{d_p} \gamma_k + (1 + r\|S^v\|_2^{1/2})(\nu + \mu + 1)\|S^v\|_2 \sum_{k=r+1}^{d_v} \lambda_k\right].$$

Space/time discretization and implementation

Space discretization: $\mathbf{P}^2 - \mathbb{P}^2$ for **LPS-ROM**.
$\mathbf{P}^2 - \mathbb{P}^1$ for **grad-div-ROM**.

- $h \approx \mathcal{O}(10^{-2})$.

OFF-ON Time discretization: Semi-implicit BDF2.

- $\Delta t \approx \mathcal{O}(10^{-3})$.

Stabilization coefficients:

Working expression [FE analysis]

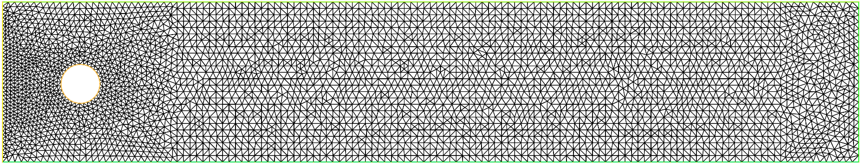
$$\tau_{\nu,K} = C_v h_K \ (C_v = 10^{-2}), \quad \tau_{p,K} = C_p h_K \ (C_p = 10^{-2}), \quad \mu = \mu_K = h_K.$$

Implementation:

- **FreeFEM** - https://freefem.org/.

2D Unsteady flow past a circular obstacle

- Computational spatial domain:
  $\Omega = \{(0,2.2)\times(0,0.41)\}\backslash\{\mathbf{x} : (\mathbf{x}-(0.2,0.2))^2 \leq 0.05^2\}$.
- Data: $\mathbf{u}(0,y,t) = (4U_m y(A-y)/A^2, 0)^{\mathrm{T}}$,
  $U_m = \mathbf{u}(0, A/2, t) = 1.5$, $\mathbf{f} = \mathbf{0}$,
  $\nu = 10^{-3}$, $\overline{U} = 2U_m/3 = 1$, $D = 0.1 \Rightarrow Re = \overline{U}D/\nu = 100$.
- Outlet boundary conditions: Do nothing.



2D Unsteady flow past a circular obstacle

Quantities of interest:

- **Kinetic energy**:

$$E_{Kin} = \frac{1}{2}\|\mathbf{u}\|^2_{\mathbf{L}^2}.$$

- **Drag coefficient**:

$$c_D = -\frac{2}{D\overline{U}^2}\left[(\partial_t \mathbf{u}, \mathbf{v}_D) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}_D) + \nu(\nabla \mathbf{u}, \nabla \mathbf{v}_D) - (p, \nabla \cdot \mathbf{v}_D)\right].$$

- **Lift coefficient**:

$$c_L = -\frac{2}{D\overline{U}^2}\left[(\partial_t \mathbf{u}, \mathbf{v}_L) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}_L) + \nu(\nabla \mathbf{u}, \nabla \mathbf{v}_L) - (p, \nabla \cdot \mathbf{v}_L)\right].$$

$\mathbf{v}_D, \mathbf{v}_L \in \mathbf{H}^1$ s.t. $\mathbf{v}_D = (1,0)^T$, $\mathbf{v}_L = (0,1)^T$ on Γ_c and null on the rest.

2D Unsteady flow past a circular obstacle

Offline phase: **LPS-FEM**, Time interval: $[0, T] = [0, 7]$.

- **Final vorticity** (Do nothing BC at the outlet).

2D Unsteady flow past a circular obstacle

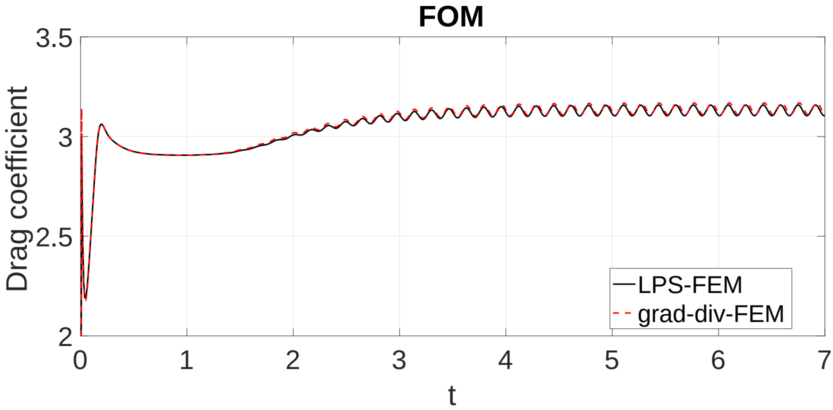
Offline phase: **grad-div-FEM**, Time interval: $[0, T] = [0, 7]$.

- **Final vorticity** (Do nothing BC at the outlet).

2D Unsteady flow past a circular obstacle

Offline phase: **LPS-FEM** *vs.* **grad-div-FEM**.

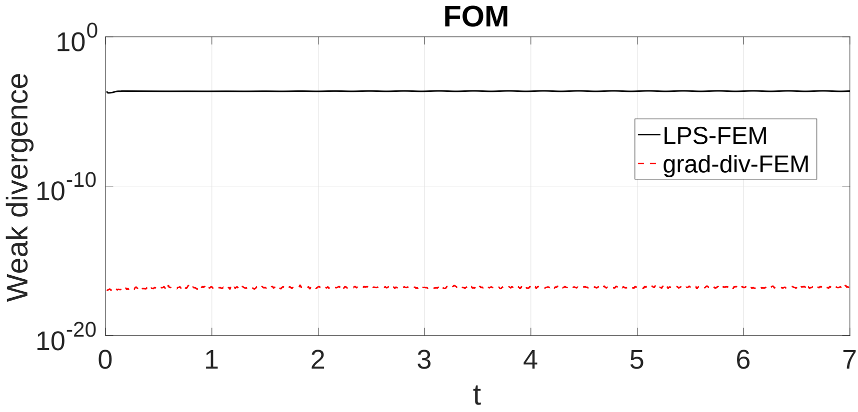
- **Drag coefficient temporal evolution**.



2D Unsteady flow past a circular obstacle

Offline phase: **LPS-FEM** *vs.* **grad-div-FEM**.

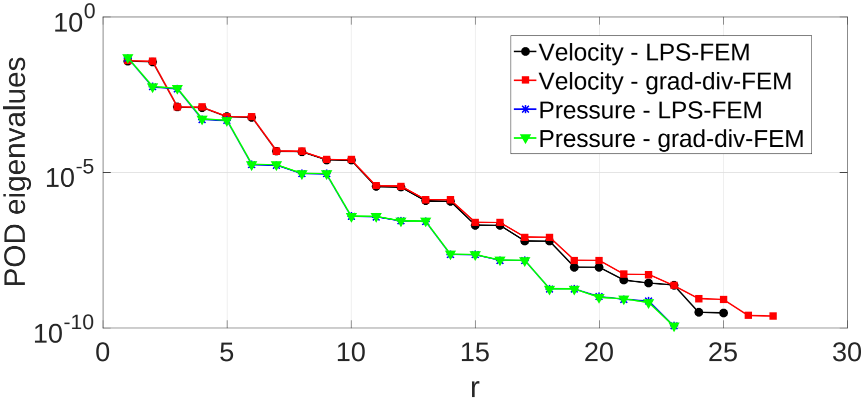
- **Weak divergence temporal evolution**.



2D Unsteady flow past a circular obstacle

Snapshots: Every **FOM** solution in $[T_s, T_p] = [5, 5.332]$.

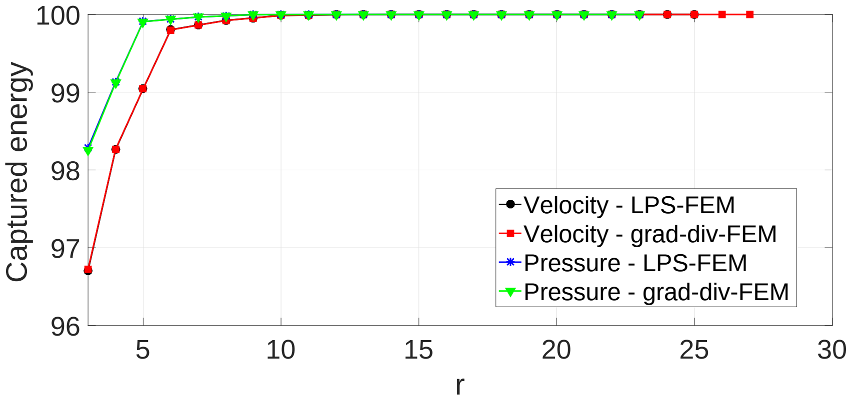
- **Decay of POD velocity and pressure eigenvalues**.



2D Unsteady flow past a circular obstacle

Snapshots: Every **FOM** solution in $[T_s, T_p] = [5, 5.332]$.

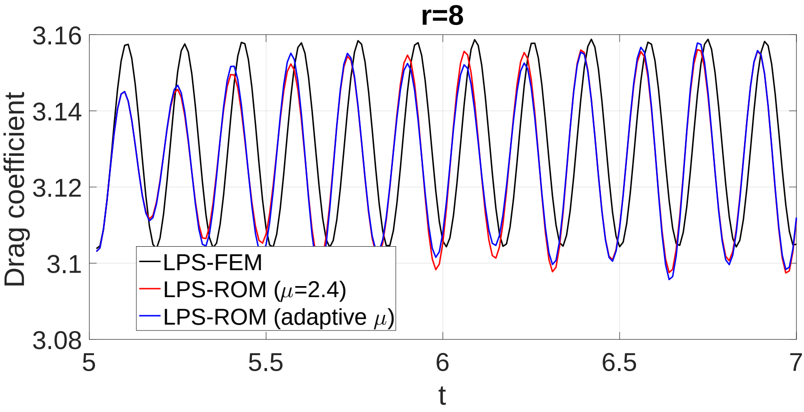
- **Captured system's energy** ($> 99\%$ for $r \geq 5$).



2D Unsteady flow past a circular obstacle

Online phase: **LPS-ROM**, Time interval: $[T_s, T] = [5, 7]$.

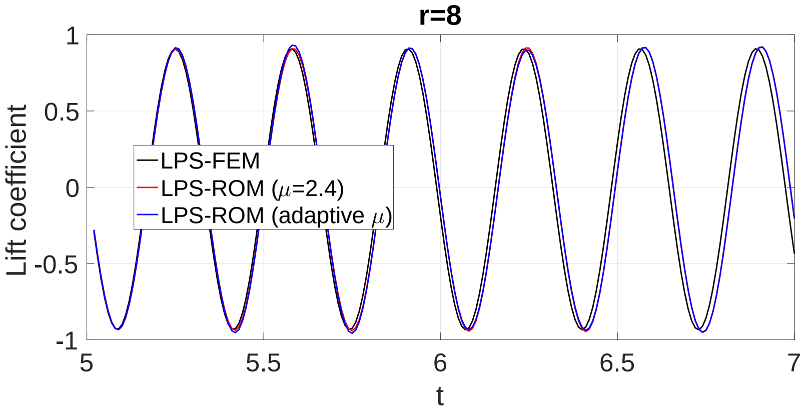
- **Drag coefficient temporal evolution**.



2D Unsteady flow past a circular obstacle

Online phase: **LPS-ROM**, Time interval: $[T_s, T] = [5, 7]$.

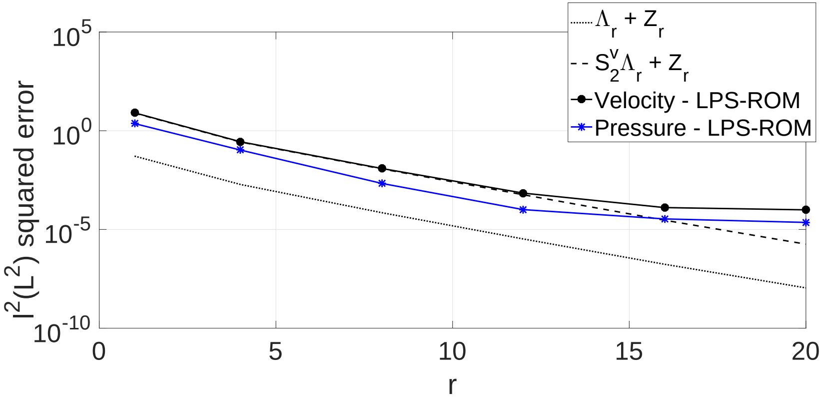
- **Lift coefficient temporal evolution**.



2D Unsteady flow past a circular obstacle

Online phase: **LPS-ROM**, Time interval: $[T_s, T] = [5, 7]$.

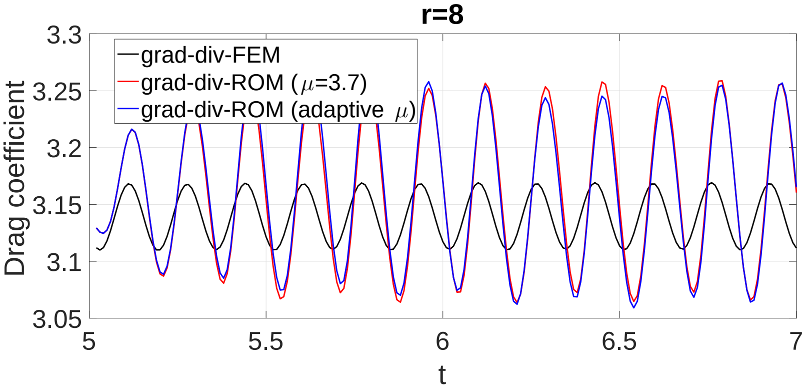
- **$\ell^2(L^2)$ vel.-pres. error w.r.t. LPS-FEM (adaptive μ)**.



2D Unsteady flow past a circular obstacle

Online phase: **grad-div-ROM**, Time interval: $[T_s, T] = [5, 7]$.

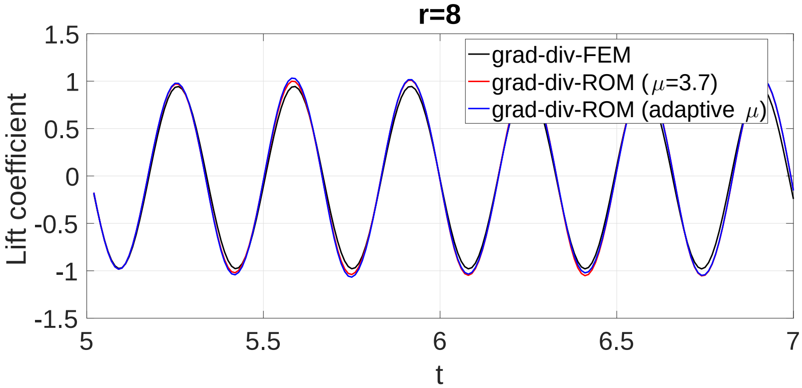
- **Drag coefficient temporal evolution**.



2D Unsteady flow past a circular obstacle

Online phase: **grad-div-ROM**, Time interval: $[T_s, T] = [5, 7]$.

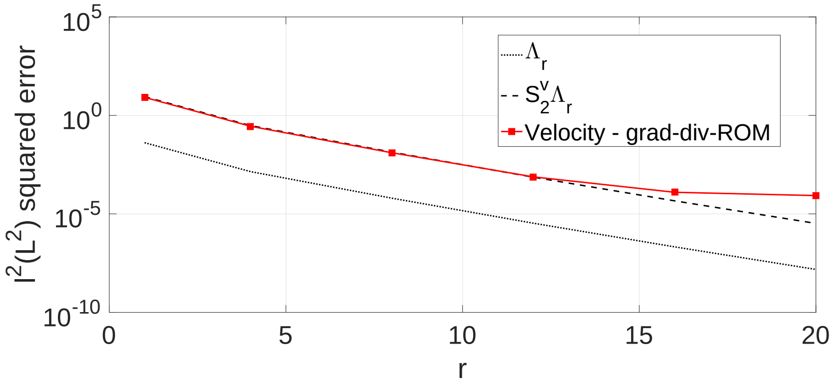
- **Lift coefficient temporal evolution**.



2D Unsteady flow past a circular obstacle

Online phase: **grad-div-ROM**, Time interval: $[T_s, T] = [5, 7]$.

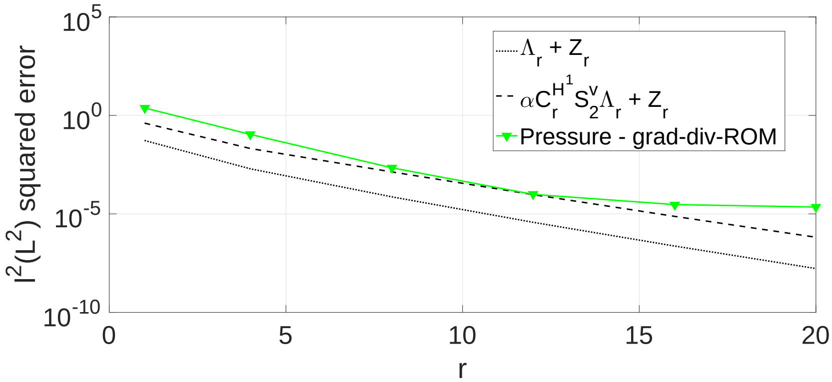
- $\ell^2(L^2)$ **velocity error w.r.t. grad-div-FEM (adaptive** μ**)**.



2D Unsteady flow past a circular obstacle

Online phase: **grad-div-ROM**, Time interval: $[T_s, T] = [5, 7]$.

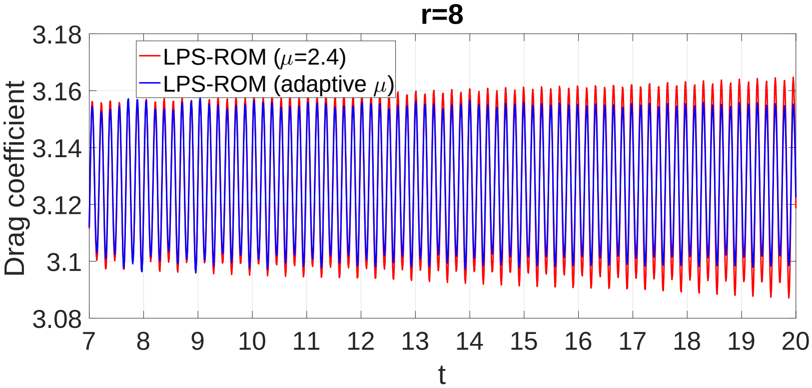
- **$\ell^2(L^2)$ pressure error w.r.t. grad-div-FEM (adaptive μ)**.



2D Unsteady flow past a circular obstacle

Online phase: **LPS-ROM**, Time interval: $[T_P, T_L] = [7, 20]$.

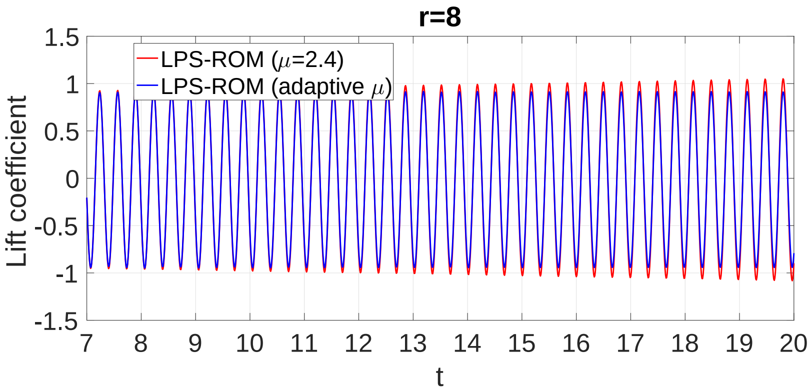
- **Drag coefficient temporal evolution** (long time behavior).



2D Unsteady flow past a circular obstacle

Online phase: **LPS-ROM**, Time interval: $[T_P, T_L] = [7, 20]$.

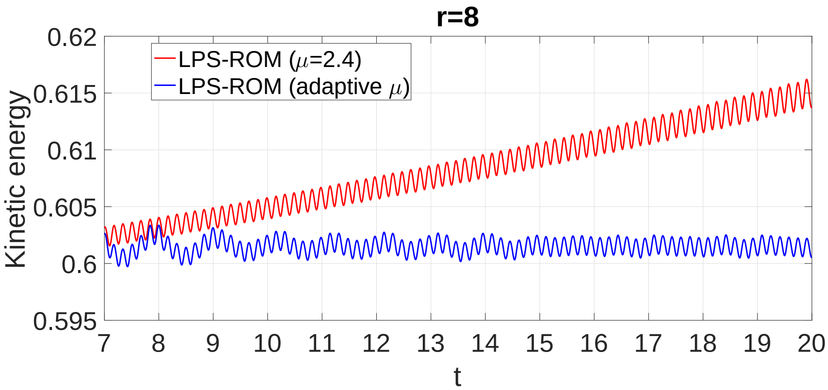
- **Lift coefficient temporal evolution** (long time behavior).



2D Unsteady flow past a circular obstacle

Online phase: **LPS-ROM**, Time interval: $[T_P, T_L] = [7, 20]$.

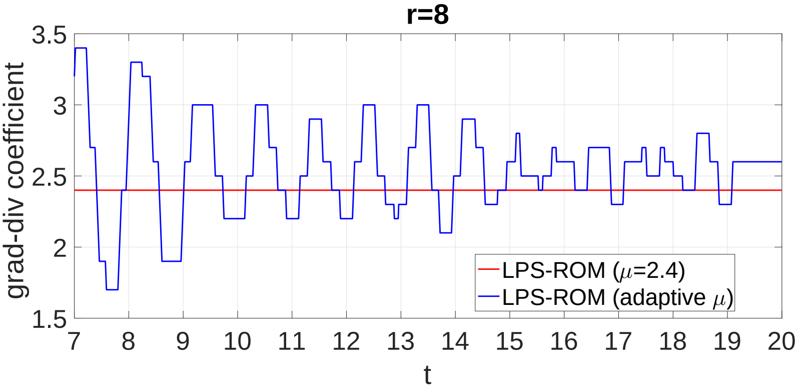
- **E_{Kin} temporal evolution** (long time behavior).



2D Unsteady flow past a circular obstacle

Online phase: **LPS-ROM**, Time interval: $[T_P, T_L] = [7, 20]$.

- μ **temporal evolution** (long time behavior).



Conclusions

Velocity-pressure LPS-ROM:

1. Accurate/efficient computation of c_D, c_L.
2. Circumvent the expensive **inf-sup** condition (NO supremizers).
3. NO **divergence-free** velocity modes required (as for PPE).
4. Numerical analysis: error bounds indep. of ν^{-1}.
5. Adaptive μ: significant improvement of long time accuracy.

Work in Progress & Future Research Lines:

- Similar numerical analysis and comparison with **SUPG-ROM**.
- Extension to *convection-dominated and turbulent flows*.

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Thank you for your kind attention!