THEORETICAL ASPECTS OF NON-LINEAR APPROXIMATION

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1. APPROXIMATION SCHEME

- X = space where approximatión takes place

  Assume $(X, \| \ \|)$ is a quasi-normed linear space

- An approximation scheme is a collection $\mathcal{G} = \{G_N\}_{N=0}^{\infty}$ of subsets of X such that $G_0 = \emptyset$, $0 \in G_N$ and:

  i) $G_N \subset G_{N+1}$; ii) $\overline{\bigcup_{N=1}^{\infty} G_N} = X$

  iii) $\exists\, C > 0$ s.t. $G_N \pm G_M \subset G_{C(N+M)}$

Best error of approximation of $f \in X$ by $\mathcal{G}$:

$$\sigma_N(f, \mathcal{G}, X) \equiv \sigma_N(f) = \inf_{g \in G_N} \| f - g \|_X$$

2. EXAMPLES

Example 1: Linear approximation

Let $\{\phi_n\}_{n=1}^{\infty} \subset X$ s.t. $\overline{\text{span}\,\{\phi_n\}_{n=1}^{\infty}} = X$, and $G_N = \text{span}\,\{\phi_n\}_{n=1}^{N}$. The sets G_N are linear and $G_N + G_M = \subset G_{max(N,M)} \subset G_{N+M}$. The problem is to find $(a_n)_{n=1}^{N}$ s.t. $\sum_{n=1}^{N} a_n \phi_n \sim f$.

Example 2: N-term approximation

Let $\{\phi_n\}_{n=1}^{\infty} \subset X$ s.t. $\overline{\text{span}\,\{\phi_n\}_{n=1}^{\infty}} = X$

$$G_N = \left\{ \sum_{j \in A} a_j \phi_j : A \subset \mathbb{N} \text{ and } \#A \leq N \right\}$$

G_N is non-linear, but $G_N + G_M \subset G_{N+M}$.

In this case the problem is to find $A_N \subset \mathbb{N}$ with $\#A_N \leq N$ and $\{a_j\}_{j=1}^{N}$ s.t. $\sum_{j \in A_N} a_j \phi_j \sim f$.

3. PROBLEM

Start with an approximation scheme $(X, \{G_N\}_{N=1}^{\infty})$.

Design a (simple) algorithm A_N that for each $f \in X$, chooses at each step an element $A_N(f) \in G_N$. Find the deviation rate of the algorithm from the best approximation; that is, find $\varphi : \mathbb{N} \to \mathbb{R}$, $\varphi(N) \geq 1$, s.t.

$$\| f - A_N(f) \|_X \leq C \varphi(N) \sigma_N(f) \quad \forall N$$

or

$$\sup_{\sigma_N(f) \neq 0} \frac{\| f - A_N(f) \|_X}{\sigma_N(f)} \approx \varphi(N) = L_N(\mathcal{B}, X) \quad \forall N$$

The best situation is attained if $\varphi(N) = 1$ for all N. In this case the algorithm is called exact for the given approximation scheme.

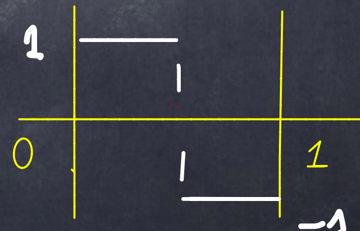
4. OUR SETTING

$\mathcal{B}=\{\phi_n\}_{n=1}^{\infty}$ basis in a Banach space $(X, \| \ \|)$ - real or complex - with biorthogonal functionals $\mathcal{B}^*=\{\phi_n^*\}_{n=1}^{\infty} \subset X^*$, that is, $\phi_n^*(\phi_m)=\delta_{n,m}$:

- Seminormalized: $0<c_1 \leq \|\phi_n\|, \|\phi_n^*\|_* \leq c_2<\infty$
- $f \sim \sum_{n=1}^{\infty} \phi_n^*(f)\phi_n$ with $|\phi_n^*(f)| \longrightarrow 0$ as $n \uparrow \infty$, $f \in X$

EXAMPLES

1. $\mathcal{B}=\{\phi_n\}_{n=1}^{\infty}$ o.n.b. in a Hilbert space
2. $\mathcal{T}^d=\{e^{i\langle x,k\rangle} : k \in \mathbb{Z}^d\}$ trigonometric system in $L^p(\mathbb{T}^d)$, $1 \leq p \leq \infty$
   ($L^\infty(\mathbb{T}^d)$ is identify with $C(\mathbb{T}^d)$ with the sup. norm)
3. Haar basis in $L^p([0,1])$ or $L^p(\mathbb{R})$



4. Wavelet bases in $L^p(\mathbb{R}^d)$, $L^p([0,1])$, Sobolev or Besov spaces

5. GREEDY ALGORITHM

$\mathcal{B} = \{\phi_n\}_{n=1}^{\infty}$ basis in $(X, \|\ \|)$

Approximation scheme : $G_N = \{ \sum_{n \in A} a_n \phi_n : A \subset \mathbb{N},\ \#A \leq N \}$

$$f = \sum_{n=1}^{\infty} \phi_n^*(f) \phi_n$$

Choose $k_1, k_2, \ldots, k_N, \cdots \in \mathbb{N}$ s.t.

$$|\phi_{k_1}^*(f)| \geq |\phi_{k_2}^*(f)| \geq \ldots \geq |\phi_{k_N}^*(f)| \geq \ldots$$

THE GREEDY ALGORITHM IS

$$\boxed{\mathcal{G}_N(f) = \sum_{j=1}^{N} \phi_{k_j}^*(f) \phi_{k_j} \in G_N}$$

6. GREEDY BASES

$\mathcal{B} = \{\phi_n\}_{n=1}^{\infty}$ is GREEDY for $(X, \| \ \|)$ if there exists $C > 1$ s.t.

$$\|f - \mathcal{G}_N(f)\| \leq C\,\sigma_N(f) \quad \forall f \in X, \ \forall N \in \mathbb{N}.$$

EXAMPLE: Orthonormal bases in a Hilbert space are greedy

- $f = \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n$ and $\|f\|^2 = \sum_{n=1}^{\infty} |\langle f, \phi_n \rangle|^2$ (Plancherel)
- In this case

$$\|f - \mathcal{G}_N(f)\| = \sigma_N(f)$$

and $\mathcal{B}$ is greedy with $C = 1$

7. CHARACTERIZATION OF GREEDY BASES

(Konyagin - Temlyakov - 1998)

$\mathcal{B}$ is greedy $\Longleftrightarrow$

$\mathcal{B}$ is unconditional and democratic

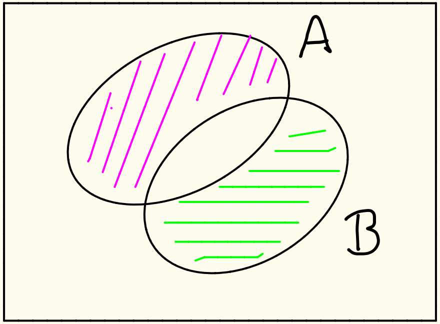
- $\mathcal{B}$ is <u>unconditional</u> if $K_u(N,\mathcal{B}) := \sup_{\#A \leq N} \|P_A\| \leq K_u \quad \forall N \in \mathbb{N}$

  where $P_A\left(\sum_{n=1}^{\infty} \phi_n^*(f)\phi_n\right) = \sum_{n \in A} \phi_n^*(f)\phi_n$

- $\mathcal{B}$ is <u>democratic</u> if $\exists\, 0 < D < \infty$ such that for all $N \in \mathbb{N}$ and all $A, B \subset \mathbb{N}$ with $\#A = \#B = N$

$$\left\|\sum_{n \in A} \phi_n\right\| \leq D \left\|\sum_{n \in B} \phi_n\right\|$$

8. $\mathcal{B}$ greedy $\Rightarrow$ $\mathcal{B}$ democratic



$\#A = \#B < \infty$. Let $\varepsilon > 0$ and

$$f = \sum_{n \in B \setminus A} (1+\varepsilon)\phi_n + \sum_{n \in A - B} \phi_n + \sum_{n \in A \cap B} \phi_n$$

Let $m = \#(B \setminus A) = \#(A \setminus B)$. Then $\mathcal{G}_m(f) = \sum_{n \in B \setminus A} (1+\varepsilon)\phi_n$ and

$$\Big\| \sum_{n \in A} \phi_n \Big\| = \| f - \mathcal{G}_m(f) \| \leq C\,\sigma_m(f) \leq C \Big\| f - \sum_{n \in A \setminus B} \phi_n \Big\|$$

$$= C \Big\| \sum_{n \in B \setminus A} (1+\varepsilon)\phi_n + \sum_{n \in A \cap B} \phi_n \Big\|.$$

Let $\varepsilon \to 0$ to obtain $\Big\| \sum_{n \in A} \phi_n \Big\| \leq C \Big\| \sum_{n \in B} \phi_n \Big\|$.

9. EXAMPLES OF GREEDY BASES

Example 1. $\mathcal{B} = \{\phi_n\}_{n=1}^{\infty} \subset X$; $B^p(\mathcal{B}, X) = \{ f \in X : \left(\sum_{n=1}^{\infty} |\phi_n^*(f)|^p\right)^{1/p} := \|f\|_p < \infty \}$

$\mathcal{B}$ is greedy for $B^p(\mathcal{B}, X)$:

- Unconditional: $\|P_A(f)\|_p = \left(\sum_{n\in A} |\phi_n^*(f)|^p\right)^{1/p} \leq \|f\|_p$
- Democratic: If $\#A = N$, $\|\sum_{n\in A} \phi_n\|_p = \left(\sum_{n\in A} 1^p\right)^{1/p} = N^{1/p}$ ind. of A.

Example 2. (Temlyakov - 1998)

The Haar basis in $L^p([0,1])$, $1<p<\infty$ is greedy

Example 3. Wavelet basis $\{\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) : j \in \mathbb{Z}, k \in \mathbb{Z}\}$ are greedy in $L^p(\mathbb{R})$, $1<p<\infty$, under mild conditions on $\psi \in L^2(\mathbb{R})$. Also greedy for Besov and Sobolev spaces.

10. THE TRIGONOMETRIC SYSTEM

$$\mathcal{T} = \{ e^{inx} ; n \in \mathbb{Z} \} \subset L^p(\mathbb{T}),\ 1 < p < \infty$$

- $\displaystyle \Big\| \sum_{|n| \le N} e^{inx} \Big\|_p = \| D_N \|_p \approx N^{1 - 1/p}$ (Dirichlet)

- For $\lambda_n = 2^n$, $n = 0, 1, 2, \ldots$, $\displaystyle \Big\| \sum_{n=0}^{2N-1} e^{i 2^n x} \Big\| \approx N^{1/2}$ (Littlewood-Paley)

Hence, $\mathcal{T}$ is democratic and greedy only in $L^p(\mathbb{T})$ only if $p = 2$

Temlyakov (1998): The deviation rate of the greedy algorithm for $\mathcal{T}$ in $L^p(\mathbb{T})$, $1 < p < \infty$, is

$$\varphi(N) \approx C N^{\left| \frac{1}{p} - \frac{1}{2} \right|}$$

11. UPPER BOUNDS FOR THE DEVIATION RATE

$$D_N(\mathcal{B}) = \sup_{\substack{\#A=N \\ \varepsilon_n=\pm 1}} \left\| \sum_{n\in A} \varepsilon_n \phi_n \right\|, \qquad D_N(\mathcal{B}^*) = \sup_{\substack{\#A=N \\ \eta_n=\pm 1}} \left\| \sum_{n\in A} \eta_n \phi_n^* \right\|.$$

$$U_N(\mathcal{B},\mathcal{B}^*) = \sum_{j=1}^{N} \frac{D_j(\mathcal{B})\, D_j(\mathcal{B}^*)}{j^2}$$

Berná-Blasco-Garrigos-Oikhberg-H-2017

$$\boxed{\varphi(N;\mathcal{B}) \leq 1 + 3\, U_N(\mathcal{B},\mathcal{B}^*)}$$

- The estimate $\varphi(N;\mathcal{T}) \leq C N^{\left|\frac{1}{p}-\frac{1}{2}\right|}$, $p\neq 2$, due to Temlyakov, can be proved using this result.

12. THE WEAK CHEBYSHEV GREEDY ALGORITHM - I

$(X, \| \ \|)$ is a Banach space

Norming functional : A normig functional for $f \in X \setminus \{0\}$ is a linear functional $F_f \in X^*$ such that $\|F_f\|_* = 1$ & $F_f(f) = \|f\|$.

$\mathcal{B} = \{\phi_n\}_{n=1}^{\infty}$ basis for $(X, \| \ \|)$

WCGA $f \in X$; start with $f_0 = f$

STEP 1 Choose $\phi_{n_1} \in \mathcal{B}$ s.t. $|F_{f_0}(\phi_{n_1})| \geq \frac{1}{2} \sup_{\phi \in \mathcal{B}} |F_{f_0}(\phi)|$

Let $CG_1(f)$ be an element of $\langle \phi_{n_1} \rangle$ such that

$$\|f - CG_1(f)\| = dist(f, \langle \phi_{n_1} \rangle).$$

Set $f_1 = f - CG(f)$.

13. THE WEAK CHEBYSHEV GREEDY ALGORITHM - II

STEP N

If $\{\phi_{n_1}, \ldots, \phi_{n_{N-1}}\} \subset \mathcal{B}$ and $\{f_1, \ldots, f_{N-2}\} \subset X$ have been choosen, let $\phi_{n_N} \in \mathcal{B}$ s.t

$$|F_{f_{N-1}}(\phi_{n_N})| \geq \frac{1}{2} \sup_{\phi \in \mathcal{B}} |F_{f_{N-1}}(\phi)|.$$

Let $\mathcal{CG}_N(f)$ be any element of $\langle \phi_{n_1}, \ldots, \phi_N \rangle$ such that

$$\| f - \mathcal{CG}_N(f) \| = \operatorname{dist}(f, \langle \phi_{n_1}, \ldots, \phi_N \rangle).$$

Set $f_N = f - \mathcal{CG}_N(f)$

Definition

Let $\varphi : \mathbb{N} \to \mathbb{N}$. A basis $\mathcal{B}$ is φ-<u>Chebyshev greedy</u> if $\exists\, C > 0$ s.t $\| f - \mathcal{CG}_{\varphi(N)}(f) \| \leq C \sigma_N(f)$ for all $f \in X$ and all $N \in \mathbb{N}$.

14. WCGA AND THE TRIGONOMETRIC SYSTEM

$\mathcal{T}^d = \{ e^{i\langle k, x\rangle} : k \in \mathbb{N} \}$ trigonometric basis in $L^p(\mathbb{T}^d)$

Temlyakov (2014)

1) If $2 \le p < \infty$, $\mathcal{T}^d$ is φ-Chevishev greedy with

$$\varphi(N) = \lceil C N \log(N+1) \rceil, \quad C = C(p,d)$$

2) If $1 < p \le 2$, $\mathcal{T}^d$ is φ-Chevishev greedy with

$$\varphi(N) = \lceil C N^{p'-1} \log(N+1) \rceil, \quad C = C(p,d)$$

Open questions: 1) Are the above assertions sharp?

2) Find an approximation algorithm for which $\varphi(N) = \lceil C N \rceil$ for the trigonometric system

15. HAAR SYSTEM

$h_0 = \mathbb{1}_{[0,1)}$ $\qquad$ $h = \mathbb{1}_{[0,\frac{1}{2})} - \mathbb{1}_{[\frac{1}{2},1)}$

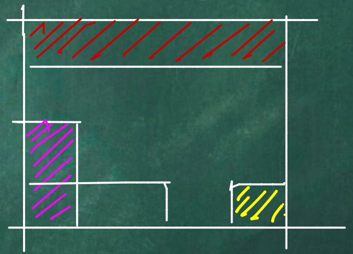
- For every dyadic interval $I = 2^{-j}[k, k+1) \in \mathcal{D}([0,1))$ define

$$h_{I,p}(x) = 2^{j/p} h(2^j x - k) \quad \text{and} \quad h_{0,p}(x) = h_0(x).$$

- If $1<p<\infty$, $\mathcal{H}^p = \{h_{0,p}, h_{I,p}\}_{I\in\mathcal{D}}$ is the L^p-normalized Haar basis, wich is unconditional in $L^p([0,1))$.

MULTIVARIATE HAAR SYSTEM

$\overline{\mathcal{D}} = \mathcal{D} \cup \{[0,1)\}$



- Let $\mathcal{R}_d$ the set of dyadic rectangles $I \subset [0,1)^d$, that is $I = I_1 \times \dots \times I_d$ with $I_j \in \overline{\mathcal{D}}$.

- For $I \in \mathcal{R}_d$, $H_{I,p}(x_1, \dots, x_d) = \prod_{j=1}^{d} h_{I_j,p}(x_j)$.

- The system $\mathcal{H}_p^d = \{H_{I,p} : I \in \mathcal{R}_d\}$ is an unconditional basis for $L^p([0,1)^d)$, $1<p<\infty$, and

$$|||f|||_p = \Big\| \Big(\sum_{I\in\mathcal{R}_d} |\langle f, H_{I,p}\rangle H_{I,p}(\cdot)|^2 \Big)^{1/2} \Big\|_{L^p([0,1)^d)} \approx \|f\|_p$$

16. WCGA FOR THE MULTIVARIATE HAAR SYSTEM

Dilworth - Garrigós - Kutzarova - Temlyakov - H (2020)

- For $1<p\leq 2$, $\mathcal{H}^d$ is φ-Chebyshev greedy with

$$\varphi(N) = \lceil C N [\log(N+1)]^{(d-1)p'\left|\frac{1}{p}-\frac{1}{2}\right|} \rceil \qquad (1)$$

- For $1<p\leq 2$, (1) is best possible
- For $2\leq p<\infty$, $\mathcal{H}^d$ is φ-Chebyshev greedy with

$$\varphi(N) = \lceil C N^{2/p'} \rceil \qquad (2)$$

- For $2\leq p<\infty$, we do not know if (2) is optimal

$X = L^p([0,1])$ and $\mathcal{B}$ greedy basis for X. Then, $\mathcal{B}$ is φ-Chebyshev greedy with

$$\varphi(N) = \begin{cases} N & \text{if } 1<p\leq 2 \\ N^{2/p'} & \text{if } 2\leq p<\infty \end{cases}$$

THE END